1. (8 pts) Find an integrating factor for the equation
\[ 2xy \, dx + (3x^2 + 2y) \, dy = 0 \]

and then solve the equation.

\[ N_x - M_y = 6x - 3x^2 = 3x(x - 2) \neq 0 \]

\[ \frac{N_x - M_y}{M} = \frac{6x}{2xy} = \frac{3}{y} = \frac{\mu'}{\mu} , \quad \mu = \mu(y) \]

\[ \int \frac{d\mu}{\mu} = \int \frac{dy}{y} \]

\[ \ln |\mu| = \ln |y| \]

\[ \mu = y \]

\[ 2xy^3 \, dx + (3x^2y^2 + 2y^3) \, dy = 0 \]

\[ M_y = 6xy^2 \]

\[ N_x = 6xy^2 \]

\[ f_x = 2xy^3 \]

\[ f_y = 3x^2y^2 + 2y^3 \]

\[ f = x^2y^3 + g(y) \]

\[ 2xy^3 + g'(y) = 3x^2y^2 + 2y^3 \]

\[ g'(y) = 2y^3 \]

\[ g(y) = \frac{y^4}{4} \]

\[ f = x^2y^3 + \frac{y^4}{4} = C \]
2. (8 pts) A turkey is at room temperature 20°C when it is put into the oven, and it is removed after 5 hours, when its internal temperature has reached 85°C. The oven temperature is 160°C. The turkey is allowed to stand outside the oven for 30 minutes prior to carving. Estimate the internal temperature of the turkey when it is carved. Assume that the transmission coefficient in the oven is the same as that outside the oven, i.e., $k$ is the same for both models.

1) \[
\frac{dT}{dt} = k(T - T_m), \quad T(0) = 20
\]
\[
T = T_m + Ce^{kt}, \quad T_m = 160
\]
\[
T(0) = 20 = 160 + Ce^0 \Rightarrow C = -140
\]
\[
T(5) = 85 = 160 - 140e^{5k} \Rightarrow e^{5k} = \frac{160 - 85}{140}
\]
\[
k = \frac{\ln\left(\frac{160 - 85}{140}\right)}{5} \approx -0.12483
\]

2) \[
\frac{dT}{dt} = k(T - T_m), \quad T(0) = 85
\]
\[
T = T_m + Ce^{kt}, \quad T_m = 20
\]
\[
T = 20 + Ce^{-0.12483t}
\]
\[
T(0) = 20 + Ce^0 = 85 \Rightarrow C = 65
\]
\[
T(0.5) = 20 + 65e^{-0.12483 \cdot 0.5} = 81°C
\]
3. (8 pts) A lake can support a population of 1000 fish. There are now 600 fish in the lake, and on the same date last year there were 300. Assuming that the logistic model determines the fish population in the lake, how many fish will be in the lake a year from now?

\[ y' = ky(1 - \frac{y}{M}) \], \quad M = 1000

\[ \int \frac{Mdy}{y(M-y)} = \int kdt \]

\[ \int \frac{(M-y+y)dy}{y(M-y)} = \int dy + \int \frac{dy}{M-y} = \int kdt \]

\[ |\frac{y}{M-y}| = kt + c \Rightarrow |\frac{y}{M-y}| = e^{kt} \Rightarrow \frac{y}{M-y} = ce^{kt} \]

\[ y(0) = 300 \quad \Rightarrow \quad \frac{y}{M-y} = Ae^{kt}, \quad A \in \mathbb{R} \]

\[ y(1) = 600 \]

\[ D = \frac{x}{3} \quad k = \ln(3.5) \]

\[ y(t) = \frac{1000e^{kt}}{\frac{5}{3} + (3.5)^t} \]

\[ y(2) = \frac{1000(3.5)^2}{\frac{5}{3} + (3.5)^2} = 840 \]

**Answer:** 840
4. (8 pts) Solve Bernoulli differential equation

\[ \frac{dy}{dx} - y = e^x y^2. \]

\[ y^{-2} y' - y = e^x \]

\[ \omega = y^{-1} \]

\[ \omega' + \omega = -e^x \]

\[ I_f = e^{\int p(x)dx} = e^{\int dx} = e^x \]

\[ e^x \omega + e^x \omega = -e^{2x} \]

\[ e^x \omega = -\int e^{2x} dx = -\frac{1}{2} e^{2x} + C \]

\[ \frac{1}{y} = -\frac{1}{2} e^x + Ce^{-x} \]

\[ y = 0 \]
5. (8 pts) Solve the initial value problem for the homogeneous differential equation

\[(x^2 + 2y^2) \frac{dx}{dy} = xy, \quad y(-1) = 1.\]

\[y = ux\]

\[(x^2 + 2ux^2) dx - ux^2(u dx + x du) = 0\]

\[x^2(1+u^2) dx - ux^3 du = 0\]

\[\frac{dx}{x} = \frac{udu}{1+u^2}\]

\[\ln|x| - \frac{1}{2} \ln(1+u^2) = C\]

\[\frac{x^2}{1+u^2} = C,\]

\[x^4 = C(x^2 + y^2)\]

\[(-1)^4 = C((-1)^2 + 1^2) = 1 = C; 2 \Rightarrow C = \frac{1}{2}\]

\[x^4 = \frac{y^2 + y^2}{2}\]