1 Linear first-order equations

A first-order linear ODE is an equation of the form

\[ a_1(t)y' + a_0(t)y = b(t), \]  

(1.1)

where \( a_1(t), a_0(t) \) are called coefficient functions, \( b(t) \) is the source term. A linear ODE is called homogeneous if the source term is 0. Every homogeneous first-order ODE can be written as

\[ y' = k(t)y, \quad k(t) := -\frac{a_0(t)}{a_1(t)} \]  

(1.2)

The general solution to (1.2) is

\[ y = Ce^{\int k(t)dt} = Ce^{-\int \frac{a_0(t)}{a_1(t)}dt} \]  

(1.3)

The general solution to inhomogeneous equation (1.1) takes the form:

\[ y = y_p(t) + Cy_h(t), \]  

(1.4)

where \( y_p(t) \) is a particular solution to (1.1) and \( y_h(t) \) is a particular nontrivial solution to corresponding homogeneous equation (1.2). In order to find (1.4) we use the method of variation of constants: substitute \( y = v(t)y_h(t) \) in (1.1) and solve for \( v(t) \).

Example.

\[ y' + 4y = 2e^{-4t}\sin(2t) \]  

(1.5)

The corresponding homogeneous equation:

\[ y' = -4y \]  

(1.6)

By formula (1.3) the general solution to (1.6) is

\[ y = Ce^{-4t} \]  

(1.7)

Now find the general solution to (1.5) in the form:

\[ y = v(t)e^{-4t} \]  

(1.8)

Substitute (1.8) into (1.5) and get

\[ v' = 2\sin(2t) \]  

(1.9)

Thus

\[ v(t) = -\cos(2t) + C \]

and therefore the general solution to (1.5) is

\[ y(t) = -e^{-4t}\cos(2t) + Ce^{-4t} \]
2 Second-order linear equations

A second-order linear ODE is an equation of the form
\[ a_2(t)y'' + a_1(t)y' + a_0(t)y = b(t) \] (2.1)

The corresponding homogeneous equation:
\[ a_2(t)y'' + a_1(t)y' + a_0(t)y = 0 \] (2.2)

The general solution to (2.2) is
\[ y_{g,h} = C_1y_1(t) + C_2y_2(t) \] (2.3)

where \( y_1, y_2 \) are two linearly independent solutions to (2.2). The general solution to inhomogeneous equation (2.1):
\[ y_{g,i} = C_1y_1(t) + C_2y_2(t) + y_p(t) \] (2.4)

where \( y_p(t) \) is a particular solution to (2.1).

I. Homogeneous equations with constant coefficients.
\[ a_2y'' + a_1y' + a_0y = 0 \] (2.5)

To solve (2.5) consider the characteristic equation:
\[ a_2s^2 + a_1s + a_0 = 0 \] (2.6)

The general solution to (2.5) is
1. \( y = C_1e^{r_1t} + C_2e^{r_2t} \), if the characteristic roots \( r_1 \) and \( r_2 \) are real and distinct.
2. \( y = e^{pt}(C_1\cos(qt) + C_2\sin(qt)) \), if the characteristic roots are complex conjugate \( p \pm qi \).
3. \( y = e^{r_1t}(C_1 + C_2t) \), if there is a double characteristic root \( r_1 \).

II. Inhomogeneous equation (2.1). If one knows \( y_1(t) \) and \( y_2(t) \), two linearly independent solutions to (2.2), then it is possible to find the general solution to (2.1) by variation of constants: substitute \( y = v_1(t)y_1(t) + v_2(t)y_2(t) \) in (2.1) and solve the following system for \( v_1'(t) \) and \( v_2'(t) \):
\[
\begin{align*}
    v_1'y_1 + v_2'y_2 &= 0 \\
    v_1'y_1' + v_2'y_2' &= b(t)/a_2(t)
\end{align*}
\]

Example. Solve the equation
\[ y'' - 2y' + y = \frac{e^x}{x} \] (2.7)
The corresponding homogeneous equation is
\[ y'' - 2y' + y = 0 \] (2.8)

The characteristic equation:
\[ s^2 - 2s + 1 = 0, \quad s_{1,2} = 1 \] (2.9)

Find the general solution to (2.7) in the form
\[ y_{g,i} = v_1(x)e^x + v_2(x)xe^x \] (2.10)

The functions \( v_1'(x) \) and \( v_2'(x) \) must satisfy the system:
\[ v_1'e^x + v_2'xe^x = 0 \]
\[ v_1'e^x + v_2'e^x(1 + x) = \frac{e^x}{x} \]

From the above system one gets
\[ v_1'(x) = -1, \quad v_2'(x) = \frac{1}{x} \]

Thus
\[ v_1(x) = -x + C_1, \quad v_2(x) = \ln |x| + C_2 \]

and
\[ y_{g,i} = (C_1 - x)e^x + (\ln |x| + C_2)xe^x = e^x(A_1 + A_2x + x \ln |x|) \] (2.11)

III. In order to solve the equation
\[ a_2y'' + a_1y' + a_0y = b(t) \] (2.12)
in some cases one can use the method of undetermined coefficients.

**Example.** Solve the equation
\[ y'' - 3y' + 2y = x \cos x \] (2.13)

The corresponding homogeneous equation is
\[ y'' - 3y' + 2y = 0 \] (2.14)

The characteristic equation:
\[ s^2 - 3s + 2 = 0, \quad s_1 = 1, \quad s_2 = 2 \] (2.15)

Therefore
\[ y_{g,h} = C_1e^x + C_2e^{2x} \] (2.16)
The right-hand side of (2.13) is \(x \cos x\), a first degree polynomial multiplied by \(\cos x\), so we are looking for the particular solution to (2.13) in the form

\[ y_p = (A_1 x + B_1) \cos x + (A_2 x + B_2) \sin x \]  

(2.17)

One has

\[ y'_p = (A_1 + B_2 + A_2 x) \cos x + (A_2 - B_1 - A_1 x) \sin x \]  

(2.18)

and

\[ y''_p = (2A_2 - B_1 - A_1 x) \cos x - (2A_1 + B_2 + A_2 x) \sin x \]  

(2.19)

Substitute \(y_p(x)\) and its derivatives in (2.13) and get the following system for \(A_1, A_2, B_1\) and \(B_2\):

\[
\begin{align*}
-3A_1 + 2A_2 + B_1 - 3B_2 &= 0 \\
A_1 - 3A_2 &= 1 \\
-2A_1 - 3A_2 + 3B_1 + B_2 &= 0 \\
3A_1 + A_2 &= 0
\end{align*}
\]

This implies

\[ A_1 = 0.1, \quad A_2 = -0.3, \quad B_1 = -0.12, \quad B_2 = -0.34 \]

Hence

\[ y_p = C_1 e^x + C_2 e^{2x} + (0.1x - 0.12) \cos x - (0.3x + 0.34) \sin x \]  

(2.20)

**IV. Reduction of order.** The reduction of order procedure is designed to find the general solution of inhomogeneous equation (2.1). It can be applied if a nontrivial solution \(y_1(t)\) of the associated homogeneous equation (2.2) is known. The procedure is as follows: substitute \(y(t) = v(t)y_1(t)\) in equation (2.1), where \(v(t)\) is a new dependent variable. When simplified, equation (2.1) becomes a first-order linear ODE with dependent variable \(w(t) := v'(t)\).

### 3 Homework

1. Solve first-order linear equations

   \[ y' + y \tan x = \sec x \quad (y = \sin x + C \cos x) \]

   \[ y = x(y' - x \cos x) \quad (y = x(C + \sin x)) \]

   \[ (xy' - 1) \ln x = 2y \quad (y = C \ln^2 x - \ln x) \]
2. Solve second-order linear equations with constant coefficients

\[ y'' - 3y' + 2y = \sin x \quad (y = C_1 e^x + C_2 e^{2x} + 0.1 \sin x + 0.3 \cos x) \]

\[ y'' - 4y' + 8y = e^{2x} + \sin 2x \]

\[(y = e^{2x}(C_1 \cos 2x + C_2 \sin 2x) + 0.25e^{2x} + 0.1 \cos 2x + 0.05 \sin 2x)\]

3. Solve second-order linear equations with variable coefficients by reduction of order

\[ x^2y'' \ln x - xy' + y = 0, \quad y_1 = x \quad (y = C_1 x + C_2 (\ln x + 1)) \]

\[ xy'' - (2x + 1)y' + (x + 1)y = 0 \quad y_1 = e^x \quad (y = e^x(C_1 x^2 + C_2)) \]

4. Solve by variation of constants

\[ y'' + 3y' + 2y = \frac{1}{e^x + 1} \quad (y = (e^{-x} + e^{-2x}) \ln(e^x + 1) + C_1 e^{-x} + C_2 e^{-2x}) \]

\[ y'' + y = \frac{1}{\sin x} \quad (y = (C_1 + \ln |\sin x|) \sin x + (C_2 - x) \cos x) \]