VECTORS CALCULUS - EXAM I

Name ______________________
Course _____________________

Find \( \mathbf{a}_T(t) \) and \( \mathbf{a}_N(t) \) at the point \((\pi, 2, \pi)\) for an object moving along the path
\[
\mathbf{r}(t) = (\pi t - \sin \pi t, 1 - \cos \pi t, \pi)
\]

\[
\mathbf{v}(t) = \mathbf{r}'(t) = (\pi - \pi \cos \pi t, \pi \sin \pi t, 0),
\]
\[
\mathbf{a}(t) = \mathbf{r}''(t) = (\pi \sin \pi t, \pi^2 \cos \pi t, 0)
\]
\[
\mathbf{a}(t_0) = (\pi t_0 - \sin \pi t_0, 1 - \cos \pi t_0, \pi) = (\pi, 2, \pi).
\]

\[
\pi t_0 - \sin \pi t_0 = \pi
\]
\[
1 - \cos \pi t_0 = 2, \ t_0 = 1 \text{ works}
\]

\[
\mathbf{v}(t) = (\pi - \pi \cos \pi t, \pi \sin \pi t, 0) = (2\pi, 0, 0)
\]
\[
\mathbf{a}(t) = (0, -\pi^2, 0)
\]
\[
\mathbf{a}_T = \frac{\mathbf{a} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = \frac{(0, -\pi^2, 0) \cdot (2\pi, 0, 0)}{(2\pi, 0, 0) \cdot (2\pi, 0, 0)} = (\pi^2, 0, 0)
\]
\[
\mathbf{a}_N = \mathbf{a} = (0, -\pi^2, 0).
\]
Given \( f(x, y) = 2 - \frac{x}{2} - \frac{y}{3} \)

(a) Sketch the graph of \( f \) in the first octant and plot the point \((2, 3, 0)\) on the surface

\[
\mathbf{z} = 2 - \frac{x}{2} - \frac{y}{3} \Rightarrow \frac{x}{2} + \frac{y}{3} + z = 2
\]

\[
\frac{x}{2} + \frac{y}{3} + \frac{z}{2} = 1
\]

(b) Find \( D_u f(2, 3) \) where \( u = \left(\frac{\sqrt{6}}{2}, -\frac{1}{2}\right) \)

\[
\nabla f = \left(-\frac{1}{2}, -\frac{1}{3}\right) \Rightarrow \nabla f (2, 3) = \left(-\frac{1}{2}, -\frac{1}{3}\right)
\]

\[
D_u f (2, 3) = \nabla f (2, 3) \cdot \mathbf{u} = \left(-\frac{1}{2}, -\frac{1}{3}\right) \cdot \left(\frac{\sqrt{6}}{2}, -\frac{1}{2}\right) = -\frac{\sqrt{6}}{4} + \frac{1}{6} = \frac{-3\sqrt{6} + 2}{12}
\]

(c) Find \( D_u f (2, 3) \) where \( u = \frac{\nabla f}{\|
abla f\|} \) and \( v \) is the vector from \((1, 2)\) to \((-2, 6)\)

\[
\nabla f = \left(-2, 4, -2\right) \Rightarrow \frac{\nabla f}{\|
abla f\|} = \frac{\left(-2, 4, -2\right)}{\|
abla f\|} = \left(-\frac{2}{\sqrt{20}}, \frac{4}{\sqrt{20}}, -\frac{2}{\sqrt{20}}\right)
\]

\[
D_u f (2, 3) = \nabla f (2, 3) \cdot \mathbf{u} = \left(-\frac{1}{2}, -\frac{1}{3}\right) \cdot \left(-\frac{2}{\sqrt{20}}, \frac{4}{\sqrt{20}}, -\frac{2}{\sqrt{20}}\right) = \frac{2}{10} - \frac{4}{15} = \frac{9 - 8}{30} = \frac{1}{30}
\]

(d) Find the maximum value of the directional derivative at \((2, 3)\)

\[
\max D_u f (2, 3) = \|
abla f (2, 3)\| = \sqrt{\frac{1}{4} + \frac{1}{9}} = \sqrt{\frac{9 + 4}{36}} = \frac{\sqrt{13}}{6}
\]
Evaluate the line integral

\[ I = \int_a^b \left( 2y \frac{dx}{dt} - 3x \frac{dy}{dt} \right) dt = \int_a^b \left( 2y(t)x'(t) - 3x(t)y'(t) \right) dt \]

for each of these three curves:

\( C_1 \): The straight line segment in the plane from A(1, 1) to B(2, 4);
\( C_2 \): The plane path from A(1, 1) to B(2, 4) along the graph of the parabola \( y = x^2 \); and
\( C_3 \): The straight line in the plane from A(1, 1) to Q(2, 1) followed by the straight line from Q(2, 1) to B(2, 4)

\[ C_1: \quad \vec{r}_1(t) = (1, 1) + t(2-1, 4-1) = (1+t, 1+3t) = (x(t), y(t)) \]
\[ \vec{r}_1'(t) = (1, 3) = (x'(t), y'(t)), \quad t \in [0, 1] \]

\[ I = \int_0^1 \left( 2(1+3t) \cdot 1 - 3(1+3t) \cdot 3 \right) dt = \int_0^1 (2+6t - 9-9t) dt = \int_0^1 (-7+3t) dt \]

\[ = - \frac{4t^2}{2} \bigg|_0^1 = -4 - \frac{3}{2} = -\frac{11}{2} \]

\[ C_2: \quad \vec{r}_2(x) = (x, x^2), \quad x \in [1, 2] \]
\[ \vec{r}_2'(x) = (1, 2x) \]

\[ I = \int_1^2 (2x^2 \cdot 1 - 3x \cdot 2x) dx = \int_1^2 (2x^2 - 6x^2) dx = -4 \int_1^2 x^2 dx = -\frac{4x^3}{3} \bigg|_1^2 \]

\[ = -\frac{4}{3} (8-1) = \frac{-28}{3} \]

\[ C_3: \quad \vec{r}_3(t) = (1, 1) + t(2-1, 0-1) = (1+t, 1) \]
\[ \vec{r}_3'(t) = (1, 0), \quad t \in [0, 1] \]

\[ \vec{r}_4(t) = (2, 1) + t(2-2, 4-1) = (2, 1+3t) \]
\[ \vec{r}_4'(t) = (0, 3), \quad t \in [0, 1] \]

\[ I_1 = \int_0^1 (2(1+3t) \cdot 1 - 3(1+3t) \cdot 0) dt = \int_0^1 dt = 1 \]
\[ I_2 = \int_0^1 (2(1+3t) \cdot 0 - 3(1+3t) \cdot 3) dt = \int_0^1 (-18 dt = -18 \]

\[ I = I_1 + I_2 = 2 + (-18) = -16 \]
Show that the given line integral is independent of path in the entire $xy$-plane, and then calculate the value of the line integral

$$\int_{(\pi/3,\pi/4)}^{(\pi/2,\pi/2)} \left( \begin{array}{c} (\sin y + y \cos x) dx + (\sin x + x \cos y) dy \\ \rho \\ \theta \end{array} \right)$$

$$\frac{\partial \rho}{\partial y} = \cos y + \cos x = -\frac{\partial \theta}{\partial x}$$

$$f_x = \sin y + y \cos x$$

$$f_y = \sin x + x \cos y$$

$$f = x \sin y + y \sin x + g(y)$$

$$f_y = x \cos y + \sin x + g'(y)$$

$$= x \cos x + x \cos y$$

$$g'(y) = 0 \Rightarrow g = C \quad \text{[take } c = 0\text{]}$$

$$\mathbf{f}(x, y) = x \sin y + y \sin x$$

$$J = f(B) - f(A) = \frac{\pi}{2} \sin \frac{\pi}{2} + \frac{\pi}{2} \sin \frac{\pi}{2} - \left( \frac{\pi}{3} \sin \frac{\pi}{4} + \frac{\pi}{4} \sin \frac{\pi}{3} \right)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} - \left( \frac{\pi}{3} \cdot \frac{1}{\sqrt{2}} + \frac{\pi}{4} \cdot \frac{\sqrt{3}}{2} \right) = \boxed{\frac{\pi}{2} - \frac{\pi}{3} \left( \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{8} \right)}$$
(MATH 6258/PHYS 6510) Compute \( \int_C x \, dx \), \( \int_C x \, dy \), \( \int_C x \, dz \), where \( C \) is the curve of intersection of the cylinder \( 2x^2 + z = 3 \) and the plane \( x = y - 1 \) from \( (0, 1, 3) \) to \( (1, 2, 1) \)

\[
\begin{align*}
x &= 3 - 2x^2 \\
y &= x + 1 \\
f_0(x) &= (x, x + 1, 3 - 2x^2) \\
f_1'(x) &= (1, 1, -6x) = (x', y', z'), \quad x \in (0, 1)
\end{align*}
\]

\[
\int_C x \, dx = \int_0^1 x \, dx = \frac{x^2}{2} \bigg|_0^1 = \frac{1}{2}
\]

\[
\int_C x \, dy = \int_0^1 (x + 1) \, dx = \frac{1}{2}
\]

\[
\int_C x \, dz = \int_0^1 (-6x) \, dx = \int_0^1 -6x^2 \, dx = -\frac{6x^3}{3} \bigg|_0^1 = -2
\]