Global organization of spiral structures in bi-parameter space of dissipative systems with Shilnikov saddle-foci

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We reveal and give a theoretical explanation for spiral-like structures of periodicity hubs in the bi-parameter space of a generic dissipative system. We show that organizing centers for “shrimp”-shaped connection regions in the spiral structure are due to the existence of L. Shilnikov homoclinics near a codimension-2 bifurcation of saddle-foci.

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Over recent years, a great deal of experimental studies and modeling simulations have been directed toward the identification of various dynamical and structural invariants to serve as the key signatures uniting often diverse nonlinear systems in a single class.

One such class of low order dissipative systems has been identified to possess one common, easily recognizable pattern involving spiral structures called the periodicity hub along with shrimp-shaped domains in a bi-parametric phase space [1, 2]. Such patterns turn out to be ubiquitously alike in both time-discrete [3, 4] and time-continuous systems [5–7], as well as in experiments [1, 8].

Despite the overwhelming number of studies reporting the occurrence of spiral structures, there is still little known about the fine construction details and underlying bifurcation scenarios for these patterns. In this Letter we study the genesis of the spiral structures in two exemplary, low order systems and reveal the generality of underlying global bifurcations. We will demonstrate that such parametric patterns along with shrimp-shaped zones are the key feature of systems with homoclinic connections involving saddle-foci meeting the single Shilnikov condition [9]. The occurrence of this bifurcation causing complex dynamics is common for a plethora of dissipative systems, describing (electro)chemical reactions [10], population dynamics [11], electronic circuits and nonlinear optics [2, 8, 12].

The first paradigmatic example is the canonical Rössler system [13]:

\[
\begin{align*}
\dot{x} &= a - (y + z), \\
\dot{y} &= x + ay, \\
\dot{z} &= b + z(x-c),
\end{align*}
\]

(1)

with two bifurcation parameters \(a\) and \(c\) (we fix \(b = 0.2\)). This classical model exhibits the spiral and screw chaotic attractors after a period doubling cascade followed by the Shilnikov bifurcation of the saddle-focus. For \(c^2 > 4ab\) the model has two equilibrium states, \(P_{1,2}(-ap_\pm,p_\pm,-p_\pm)\), where \(p_\pm = (-c \mp \sqrt{c^2 - 4ab})/2a\).

The second example is the Rosenzweig-MacArthur tritrophic food chain model [11, 14]:

\[
\begin{align*}
\dot{x} &= x \left[ r \left( 1 - x/K \right) - 5y/(1 + 3z) \right], \\
\dot{y} &= y \left[ 5y/(1 + 3x) - z/10(1 + 2y) - 0.4 \right], \\
\dot{z} &= z \left[ y/10(1 + 2y) - 0.01 \right],
\end{align*}
\]

(2)

for a food chain composed of a logistic prey, \(x\), a Holling type II predator, \(y\), and a top-predator, \(z\); two bifurcation parameters \(K\) and \(r\) controls the regrowth rates of the prey [11].

Bi-parametric screening the Rössler (panels A-B) and food chain (panels C-D) models unveils a stunning universality of the periodicity hubs in the bifurcation diagrams shown in Fig. 1 of both models. Each is built on a dense grid of 1000 \(\times\) 1000 points in the parameter plane. Solutions of the models were integrated using a high precision ODE solver TIDES [15]. The color bars on the right in Fig. 1 yield a spectrum of the Lyapunov exponents. Panels (A) and (C) of the figure reveal the characteristic spiral patterns, where dark and light colors discriminate the regions of regular and chaotic dynamics corresponding to zero and positive Lyapunov exponent \(\lambda_1\), respectively. Panels (B) and (D) show the enhanced fine structures of the bifurcation diagrams of the models, due to variations of both Lyapunov exponents \(\lambda_1\) and \(\lambda_2\). The white stripes expose shrimp-shaped areas (within red-boxes) on the dark background of the regular (\(\lambda_1 = 0\)) region, as well as in the multicolored region corresponding to complex dynamics (\(\lambda_1 > 0\)).

Panels are overlaid with the (blue) curves, obtained using [16], corresponding to saddle-node (or fold) bifurcations of periodic orbits. The curves on-side demarcate the stability windows from chaotic regions within the spiral structure either via the intermittency of type I boundary crisis [6], or due to a classical period doubling cascade. In the case of the Rössler model, the saddle-node curves spiral onto a single T(terminal) point. This T-point also terminates a bifurcation curve (black) corresponding to a formation of a homoclinic loop of a saddle-focus, \(P_2\) of the Rössler model. Besides, another curve (green) passes though the T-point: crossing over it makes the chaotic
Figure 1. (Color online) Spirals and “shrimps” in biparametric bifurcation diagrams for the Rössler (A-B) and tritrophic food chain (C-D) models. The T-point of the hub is located at \((a, c) = (0.1798, 10.3084)\) and \((K, r^*) = (1.0587, -1.6285 \cdot 10^{-3})\) (resp.). The color bars for the Lyapunov exponent range, red for positive and black for negative values, identify the regions of chaotic and regular dynamics (resp.). For visibility the parameter plane of the food chain model is untwisted by transformation \(r^* = r + 0.11(K - 1)/0.14 - 0.83\). Left monochrome panels are superimposed with bifurcation curves: blue for saddle-nodes, and black for homoclinic bifurcations of saddle-foci. The green boundary determines a change in the topological structure of chaotic attractors from spiral (at •) to screw-shaped (at ⋆).

attractor in the phase space of the model change the topological structure from spiral to screw-shaped.

The topological structure of the Rössler attractor is described in terms of a paper-sheet model, called a template made of “normal” and twisted, alike a Möbius band, stripes. The topological model can be quantified by a set of linking numbers – the local torsions. The torsions are, locally, the crossings number of the stripes in the template, i.e. the number of twists of the layering graph between any two unstable periodic orbits in the chaotic attractor. The local torsions determine the linking matrices and hence the template of the attractor [17]. Practically, the template may be derived using a Poincaré return map defined on successive local maxima, \(y(i)\), of trajectories in the chaotic attractor and studying the unstable periodic orbits foliating the attractor. So, the spiral attractor of the Rössler model at \(a = 0.14\) (labeled by • in Fig. 1(A,C)) generates a 1D unimodal map as one shown in Fig. 2) The single critical determines the boundary between the normal and twisted (resp.) stripes. This lets a symbolic description be naturally introduced using two symbols, 0 and 1, for each branch or stripe. In the case of screw attractor at \(a = 0.18\) (labeled by ⋆ in Fig. 1(A,C)), the corresponding map in Fig. 2 has a bimodal graph, i.e. three monotone branches divided by two critical points. Hence, the symbolic dynamics here is defined on a set of three symbols: \(\{0, 1, 2\}\). Addition of the second critical point in the map is a direct indication that the spiral attractor changes its topology. This criteria was used to locate the corresponding boundary (green) that separates the existence regions of the attractors of both types in bifurcation diagram in Fig. 1. Notice that this boundary goes through the T-point.

The linking matrices, which contain all topological in-
formation for the spiral and screw attractors, are given by [18]:

$$M_{sp} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}, \quad M_{sc} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -1 & -2 \\ -1 & -2 & -2 \end{pmatrix}. \quad (3)$$

The diagonal elements in each matrix are the sum of the signed half-twists in each branch, i.e. the self-torsion. The off-diagonal elements are the sum of the oriented crossings between the branches. Thus, in the spiral attractor we have a 0 value implying that the right branch (0) has no torsion, and the middle branch (1) (left branch (2)) has a half-twist (the value $-1$ (or $-2$)). The other $-1$ values indicate that two branches cross once, as shown in the topological template.

The (black) bifurcation curve in Fig. 1(A) corresponds to a formation of the primary homoclinic orbit to the saddle-focus, $P_2$, of topological type (1,2), i.e. a 1D stable and 2D unstable manifolds, in the model (1). Depending on the magnitudes of the eigenvalues of the saddle-focus, the homoclinic bifurcation can give rise to the onset of either rich complex or trivial dynamics in the system [9, 19]. The cases under considerations meet the Shilnikov conditions and hence the existence of a single homoclinic orbit implies chaotic dynamics in the models within the parameter range in the diagrams. Magnification of the corresponding bifurcation curve the diagram reveals that what appears to be as a single bifurcation curve (black), indeed is made of two branches (Fig. 3). This curve has a U-shape with a cusp at the T-point. To examine the U-shape we plot the bifurcation curve in terms the $L_2$-norm of the homoclinic orbit against the bifurcation values of the parameter $a$. Fig. 3 shows that the T-point terminates two branches of homoclinic loops: bottom one corresponding to the primary loop, while the top one correspond to a secondary loop with two rounds.

Fig. 4 outlines the bifurcation unfolding around the spiral hub [7]. Inset (A) for the Rössler model depicts a number of identified, saddle-node bifurcation curves originating from a codimension-2 point, labeled as B(elyakov), toward the spiral hub. At this B-point, the equilibrium state, being a saddle with double real positive eigenvalues becomes a saddle-focus for smaller values of the parameter $a$. The unfolding of this bifurcation is known [20, 21] to contain bundles of countable many curves corresponding to saddle-node and period doubling bifurcations of periodic orbits, as well as various homoclinic ones. Indeed, both B- and T-points together globally determine the structure of the $(c,a)$-bifurcation portrait of the Rössler model. Fig.4(B) sketches a phenomenological scheme of the formation of the spiral hub along with the “shrimps”. All saddle-node bifurcation curves originating from the B-point determine the boundaries of different “shrimps” near the spiral hub. Indeed, the spiral structure can generate an infinite chain of “shrimps” [2, 10]. Fig.4(C) presents a few
visible shrimps, $S_{2j}$ and $S_{2j+1}$, singled out by solid red curves, also corresponding to saddle-node orbits, which fold around the T-point in the region of spiral chaotic attractor. Meanwhile, the cusp-shaped (blue) saddle-node bifurcation curves join the successive $S_{2j-1}$-th and $S_{2j}$-th in the region of the screw-type attractor. These folded and cusp-shaped bifurcation curves of saddle-node periodic orbits shape the structure of the spiral hub and of the “shrimps.” The “shrimps” serve as connection centers that form the characteristic spiral structure in the parameter plane.

We have presented a generic scenario for the formation of the spiral structures and ”shrimps” in the biparameter space of systems with Shilnikov saddle-foci. The skeleton of the structure is formed by bifurcation curves of saddle-node periodic orbits that accompany the homoclinics of the saddle-focus. These bifurcation curves distinctively shape the “shrimp” zones in the vicinity of the spiral hub. In the R"ossler model, these bifurcation curves originate from a codimension-2 Belyakov point corresponding to the transition to the saddle-focus from a simple saddle. The feature of the spiral hub in the R"ossler and the tritrophic food chain models is that the T-point gives rise to the alternation of the topology structure of the chaotic attractor transitioning between the spiral and screw-like types. The findings let us hypothesize about a universality of the structure of the spiral hubs in similar systems with chaotic attractors due to homoclinics of the Shilnikov saddle-focus.

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**Figure 4.** (Color online) Outline of the spiral structures: (A) two kinds, folded and cusp-shaped of saddle-node bifurcation curves for the R"ossler model originating from the cod-2 homoclinic B-point. (B) phenomenological sketch of the spiral hub formed by the “shrimps.” (C) Magnification of the bifurcation portrait of the spiral hub, overlaid with principal folded (red) and cusp-shaped (blue) bifurcation curves setting the boundaries for largest “shrimps” in the R"ossler model at $b = 2$.

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