§3.4 Q2

From observation of the equi-distant function values in both \( x \) \& \( y \) and the end-point derivatives pointing along the straight line formed by the \((x, y)\) data pairs, the best fit is indeed the straight line \( S(x) = x \). Now, to prove this using Theorem 3.12.

There are three data points at \( x_0 = 0, x_1 = 1, x_2 = 2 \)
\[ a_0 = f(0) = 0, \quad a_1 = f(1) = 1, \quad a_2 = f(2) = 2 \]
So \( n = 2 \) and \( h_0 = x_2 - x_1 = x_1 - x_0 = 1 \).

The asymmetry in the clamped spline case gives \( f'(x_0) = b_0 \)
so that \( b_0 = 1 \) from the given data, but \( b_1 \) doesn't come directly from \( f'(x_2) = 1 \).

Note, each spline \((j = 0, 1)\) is \( S_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3 \)

So \( f'(x_2) = 1 \) condition gives us
\[ S'_{n-1}(x_2) = 1 \quad \text{i.e.} \quad S'_1(x_2) = 1 \]
\[ \Rightarrow b_0 + 2c_1 + 3d_1 = 1 \quad \Rightarrow b_1 + 2c_1 + 3d_1 = 1 \]

But the theorem already incorporates this relationship into the three equations we need to solve:

\[ 2hc_0 + hc_1 = \frac{3}{h}(a_1 - a_0) - 3f'(x_0) \quad (1) \]
\[ hc_1 + 2hc_2 = 3f'(x_2) - \frac{3}{h}(a_2 - a_1) \quad (2) \]
and (Eq. (3,21)) \[ hc_0 + 2(h+h)c_1 + hc_2 = \frac{3}{h}(a_2-a_0) - \frac{3}{h}(a_1-a_0) \]

(only one equation as \( j = 0 \ldots n-2 \))

\[ \Rightarrow \quad c_0 + 4c_1 + c_2 = 3a_2 - 6a_1 + 3a_0 = 6 - 6 + 3(0) = 0 \]  \hspace{1cm} (3)

Eqs. (1) and (2) become (with \( h=1 \))

\[ 2c_0 + c_1 = 3 - 3 = 0 \quad \Rightarrow \quad c_1 = -2c_0 \]

\[ c_1 + 2c_2 = 3 - 3(1) = 0 \quad \Rightarrow \quad c_1 = -2c_2 \]

So \( c_2 = \frac{-c_1}{2} = c_0 \)

Subst. into (3) \[ c_0 - 8c_0 + c_0 = 0 \quad \Rightarrow \quad c_0 = 0 \]

\[ \Rightarrow c_1 = c_2 = 0 \]

Eq. (3.17): \[ c_1 = c_0 + 3d_0 h \quad \Rightarrow \quad 0 = 0 + 3d_0 \quad \Rightarrow \quad d_0 = 0 \]

\[ c_2 = c_1 + 3d_1 h \quad \Rightarrow \quad 0 = 0 + 3d_1 \quad \Rightarrow \quad d_1 = 0 \]

From \( b_0 = 1 \) and Eq. (3.16) we get

\[ b_1 = b_0 + 2c_0 h + 3d_0 h^2 \]

\[ = 1 + 0 + 0 = 1. \]

First spline: \[ S_0(x) = a_0 + b_0 (x-x_0) + c_0 (x-x_0)^2 + d_0 (x-x_0)^3 \]

\[ = 0 + 1(x-0) + 0 + 0 \quad = x \]

Second spline: \[ S_1(x) = a_1 + b_1 (x-x_1) + c_1 (x-x_1)^2 + d_1 (x-x_1)^3 \]

\[ = 1 + 1(x-1) + 0 + 0 \quad = 1 + x - 1 = x \]