whereas the approximation in Exercise 9 has
\[ \sum_{j=0}^{10} (f(\xi_j) - S_2(\xi_j))^2 = 369.3589. \]

15. Show that the functions \( \phi_0(x) = \left( \frac{1}{2} \right), \phi_1(x) = \cos x, \ldots, \phi_n(x) = \cos nx, \phi_{n+1}(x) = \sin x, \ldots, \phi_{2n-1}(x) = \sin (n - 1)x \) are orthogonal on \([-\pi, \pi]\) with respect to \( w(x) = 1 \).

SOLUTION: The following integrations establish the orthogonality.

\[ \int_{-\pi}^{\pi} \phi_k(x)^2 \, dx = \frac{1}{2} \int_{-\pi}^{\pi} dx = \pi, \]

\[ \int_{-\pi}^{\pi} \phi_k(x) \phi_{k+c}(x) \, dx \]

\[ = \pi - \left[ \frac{1}{4k} \sin 2kx \right]_{-\pi}^{\pi} = \pi, \]

\[ \int_{-\pi}^{\pi} \phi_n(x) \phi_0(x) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos kx \, dx = \frac{1}{2k} \sin kx \bigg|_{-\pi}^{\pi} = 0, \]

\[ \int_{-\pi}^{\pi} \phi_n(x) \phi_{n+c}(x) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin kx \, dx = \frac{1}{2k} \cos kx \bigg|_{\pi}^{\pi} = 0, \]

\[ \int_{-\pi}^{\pi} \phi_k(x) \phi_k(x) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos kx \cos jx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} [\cos(k + j)x + \cos(k - j)x] \, dx = 0, \]

\[ \int_{-\pi}^{\pi} \phi_n(x) \phi_n(x) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin kx \sin jx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} [\cos(k + j)x - \cos(k - j)x] \, dx = 0, \]

and

\[ \int_{-\pi}^{\pi} \phi_k(x) \phi_{k+j}(x) \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos kx \sin jx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} [\sin(k + j)x - \sin(k - j)x] \, dx = 0. \]