Chapter 5
Discrete Probability Distribution

Learning objectives

1. Understand random variables and probability distributions.
   1.1. Distinguish discrete and continuous random variables.

2. Able to compute Expected value and Variance of discrete random variable.

3. Understand:
   3.1. Discrete uniform distribution
   3.2. Binomial distribution
   3.3. Poisson distribution
A random variable is a numerical description of the outcome of an experiment.

A discrete random variable may assume either a finite number of values or an infinite sequence of values.

A continuous random variable may assume any numerical value in an interval or collection of intervals.

Example: JSL Appliances

- Discrete random variable with a finite number of values
  
  Let \( x \) = number of TVs sold at the store in one day, where \( x \) can take on 5 values (0, 1, 2, 3, 4)

- Discrete random variable with an infinite sequence of values
  
  Let \( x \) = number of customers arriving in one day, where \( x \) can take on the values 0, 1, 2, . . .

  • We can count the customers arriving, but there is no finite upper limit on the number that might arrive.
Random Variables

<table>
<thead>
<tr>
<th>Question</th>
<th>Random Variable $x$</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Family size</td>
<td>$x =$ Number of dependents reported on tax return</td>
<td>Discrete</td>
</tr>
<tr>
<td>Distance from home to store</td>
<td>$x =$ Distance in miles from home to the store site</td>
<td>Continuous</td>
</tr>
<tr>
<td>Own dog or cat</td>
<td>$x =$ 1 if own no pet; = 2 if own dog(s) only; = 3 if own cat(s) only; = 4 if own dog(s) and cat(s)</td>
<td>Discrete</td>
</tr>
</tbody>
</table>

Discrete Probability Distributions

The probability distribution for a random variable describes how probabilities are distributed over the values of the random variable.

We can describe a discrete probability distribution with a table, graph, or equation.
The probability distribution for discrete random variable is defined by a probability function, denoted by \( f(x) \), which provides the probability for each value of the random variable.

The required conditions for a discrete probability function are:

\[
\begin{align*}
  f(x) &\geq 0 \\
  \sum f(x) &= 1
\end{align*}
\]

Using past data on TV sales, ... 
A tabular representation of the probability distribution for TV sales was developed.

<table>
<thead>
<tr>
<th>Units Sold</th>
<th>Number of Days</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>80</td>
<td>0</td>
<td>.40</td>
</tr>
<tr>
<td>1</td>
<td>50</td>
<td>1</td>
<td>.25</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>2</td>
<td>.20</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>3</td>
<td>.05</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>4</td>
<td>.10</td>
</tr>
</tbody>
</table>

\( \sum f(x) = 1 \)
Discrete Probability Distributions

Graphical Representation of Probability Distribution

Expected Value and Variance

The expected value, or mean, of a random variable is a measure of its central location.

\[ E(x) = \mu = \sum x f(x) \]

The variance summarizes the variability in the values of a random variable.

\[ \text{Var}(x) = \sigma^2 = \sum (x - \mu)^2 f(x) \]

The standard deviation, \( \sigma \), is defined as the positive square root of the variance.
Expected Value and Variance

**Expected Value**

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>xf(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.40</td>
<td>.00</td>
</tr>
<tr>
<td>1</td>
<td>.25</td>
<td>.25</td>
</tr>
<tr>
<td>2</td>
<td>.20</td>
<td>.40</td>
</tr>
<tr>
<td>3</td>
<td>.05</td>
<td>.15</td>
</tr>
<tr>
<td>4</td>
<td>.10</td>
<td>.40</td>
</tr>
</tbody>
</table>

E(x) = 1.20

expected number of TVs sold in a day

**Variance and Standard Deviation**

<table>
<thead>
<tr>
<th>x</th>
<th>x - μ</th>
<th>(x - μ)^2</th>
<th>f(x)</th>
<th>(x - μ)^2f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1.2</td>
<td>1.44</td>
<td>.40</td>
<td>.576</td>
</tr>
<tr>
<td>1</td>
<td>-0.2</td>
<td>0.04</td>
<td>.25</td>
<td>.010</td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
<td>0.64</td>
<td>.20</td>
<td>.128</td>
</tr>
<tr>
<td>3</td>
<td>1.8</td>
<td>3.24</td>
<td>.05</td>
<td>.162</td>
</tr>
<tr>
<td>4</td>
<td>2.8</td>
<td>7.84</td>
<td>.10</td>
<td>.784</td>
</tr>
</tbody>
</table>

Variance of daily sales = \(\sigma^2 = 1.660\)

Standard deviation of daily sales = 1.2884 TVs
In-class Exercise

- Random variables
  - #2 (page 188)
  - #5 (page 188)

- Expected value and variance
  - #16 (page 196)
  - #17 (page 197)

Discrete Uniform Probability Distribution

The discrete uniform probability distribution is the simplest example of a discrete probability distribution given by a formula.

The discrete uniform probability function is

\[ f(x) = \frac{1}{n} \]

where:

- \( n \) = the number of values the random variable may assume

the values of the random variable are equally likely
Binomial Distribution

Four Properties of a Binomial Experiment

1. The experiment consists of a sequence of \( n \) identical trials.
2. Two outcomes, success and failure, are possible on each trial.
3. The probability of a success, denoted by \( p \), does not change from trial to trial.
4. The trials are independent.

Our interest is in the number of successes occurring in the \( n \) trials.

We let \( x \) denote the number of successes occurring in the \( n \) trials.
Binomial Distribution

- Binomial Probability Function

\[
f(x) = \binom{n}{x} p^x (1-p)^{n-x}
\]

where:

- \( f(x) \) = the probability of \( x \) successes in \( n \) trials
- \( n \) = the number of trials
- \( p \) = the probability of success on any one trial
Example: Evans Electronics

Evans is concerned about a low retention rate for employees. In recent years, management has seen a turnover of 10% of the hourly employees annually. Thus, for any hourly employee chosen at random, management estimates a probability of 0.1 that the person will not be with the company next year.

Using the Binomial Probability Function
Choosing 3 hourly employees at random, what is the probability that 1 of them will leave the company this year?

Let: \( p = 0.10, \ n = 3, \ x = 1 \)

\[
\begin{aligned}
\text{Let: } \ p &= .10, \ n = 3, \ x = 1 \\
f(x) &= \frac{n!}{x!(n-x)!} \cdot p^x (1 - p)^{n-x} \\
f(1) &= \frac{3!}{1!(3-1)!} \cdot (0.1)^1 (0.9)^2 = 3(0.1)(0.81) = 0.243
\end{aligned}
\]
Example: Evans Electronics

- Random variables
- EV and Variance
- Uniform discrete dist.
- Binomial dist.
- Poisson dist.

### Tree Diagram

1st Worker
- Leaves (.1)
- Stays (.9)

2nd Worker
- Leaves (.1)
- Stays (.9)

3rd Worker
- Leaves (.1)
- Stays (.9)

<table>
<thead>
<tr>
<th>X</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.0010</td>
</tr>
<tr>
<td>1</td>
<td>.0090</td>
</tr>
<tr>
<td>2</td>
<td>.0090</td>
</tr>
<tr>
<td>3</td>
<td>.7290</td>
</tr>
</tbody>
</table>

Example: Evans Electronics

- Using Tables of Binomial Probabilities

<table>
<thead>
<tr>
<th>n</th>
<th>x</th>
<th>.05</th>
<th>.10</th>
<th>.15</th>
<th>.20</th>
<th>.25</th>
<th>.30</th>
<th>.35</th>
<th>.40</th>
<th>.45</th>
<th>.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>.8574</td>
<td>.7290</td>
<td>.6141</td>
<td>.5120</td>
<td>.4219</td>
<td>.3430</td>
<td>.2746</td>
<td>.2160</td>
<td>.1664</td>
<td>.1250</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>.1354</td>
<td>.2430</td>
<td>.3251</td>
<td>.3840</td>
<td>.4219</td>
<td>.4410</td>
<td>.4436</td>
<td>.4320</td>
<td>.4084</td>
<td>.3750</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>.0071</td>
<td>.0270</td>
<td>.0574</td>
<td>.0960</td>
<td>.1406</td>
<td>.1890</td>
<td>.2389</td>
<td>.2880</td>
<td>.3341</td>
<td>.3750</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>.0001</td>
<td>.0010</td>
<td>.0034</td>
<td>.0080</td>
<td>.0156</td>
<td>.0270</td>
<td>.0429</td>
<td>.0640</td>
<td>.0911</td>
<td>.1250</td>
</tr>
</tbody>
</table>
**Binomial Distribution**

- **Expected Value**
  \[ E(x) = \mu = np \]

- **Variance**
  \[ \text{Var}(x) = \sigma^2 = np(1 - p) \]

- **Standard Deviation**
  \[ \sigma = \sqrt{np(1 - p)} \]

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**Example: Evans Electronics**

- **Expected Value**
  \[ E(x) = \mu = 3(.1) = 3 \text{ employees out of 3} \]

- **Variance**
  \[ \text{Var}(x) = \sigma^2 = 3(.1)(.9) = .27 \]

- **Standard Deviation**
  \[ \sigma = \sqrt{3(.1)(.9)} = .52 \text{ employees} \]
In-class Exercise

- #26 (page 207)
- #37 (page 208)

Poisson Distribution

- A Poisson distributed random variable is often useful in estimating the number of occurrences over a specified interval of time or space.
- It is a discrete random variable that may assume an infinite sequence of values ($x = 0, 1, 2, \ldots$).
Examples of a Poisson distributed random variable:
- the number of knotholes in 14 linear feet of pine board
- the number of vehicles arriving at a toll booth in one hour

Two Properties of a Poisson Experiment

1. The probability of an occurrence is the same for any two intervals of equal length.
2. The occurrence or nonoccurrence in any interval is independent of the occurrence or nonoccurrence in any other interval.
Poisson Distribution

- Poisson Probability Function

\[ f(x) = \frac{\mu^x e^{-\mu}}{x!} \]

where:
- \( f(x) \) = probability of \( x \) occurrences in an interval
- \( \mu \) = mean number of occurrences in an interval
- \( e \approx 2.71828 \)

Example: Mercy Hospital

Patients arrive at the emergency room of Mercy Hospital at the average rate of 6 per hour on weekend evenings. What is the probability of 4 arrivals in 30 minutes on a weekend evening?
Example: Mercy Hospital

Using the Poisson Probability Function

\[ \mu = 6/ \text{hour} = 3/ \text{half-hour}, \; x = 4 \]

\[ f(4) = \frac{3^4(2.71828)^{-3}}{4!} = 0.1680 \]

Example: Mercy Hospital

Using Poisson Probability Tables
Example: Mercy Hospital

Poisson Distribution of Arrivals

Number of Arrivals in 30 Minutes

Probability

0.0 1 2 3 4 5 6 7 8 9 10

Poisson Probabilities

0.25

0.20

0.15

0.10

0.05

0.00

actual, the sequence continues: 11, 12, ...

Poisson Distribution

A property of the Poisson distribution is that the mean and variance are equal.

\[ \mu = \sigma^2 \]
Example: Mercy Hospital

- Variance for Number of Arrivals During 30-Minute Periods

\[ \mu = \sigma^2 = 3 \]

In-class Exercise

- #38 (page 211)
- #41 (page 212)
End of Chapter 5