Beau Jarble's Maximization Problem

Beau Jarble has a small plant in which he makes a very limited line of inexpensive "mag" wheels for cars in the Secret Car Club of America (SCCA) Impound Touring category. All his wheels are 7" wide, and are drilled for either 4 or 5 wheel studs (for attaching the wheels). He purchases "blank" wheels cast to his design by a local foundry and finishes them in his little plant on weekends off from his regular job, using part time workers for labor. His part timers are all first rate machinists who earn $30 an hour.

The ITA model is made from a 13" diameter by 7" wide "A" casting and sells for $125. The ITS model, for heavier and more powerful cars, is made from the 14" by 7" "S" casting and sells for $145. His present stock of castings cost him $50 each for the A units and $57.50 each for the S units, but due to a drop in the price of aluminum he can get his next batch at $45 and $50. He has a good stock of castings on hand and considers the foundry to be utterly reliable. He does not anticipate being limited by the casting supply.

His wheels are highly regarded by both real racers and "cafe racers". While demand is certainly not unlimited, he figures he could sell all the ITA wheels he could possibly make. He does not think that he could sell more than 25 sets of ITS wheels per week. On the other hand, to maintain the prestige of his product, he never wants to sell more ITA than ITS.

The first operation in making a casting into a wheel is to use a milling machine to put a properly aligned flat surface on the back of the wheel where it mounts to the car. He has 3 milling machines and 3 operators who each work up to 10 hours per weekend. They are each able to "backface" 5 wheels per hour.

The next step is to drill the wheel mounting holes on a special drill press. An ITA wheel requires 4 holes, and an ITS wheel needs 5. His drill press operator can drill 60 holes/hour and is willing to work as many as 10 hours in a weekend. Once the wheels have their mounting holes drilled, they can go on a lathe to be trued and surface finished. Beau has 3 lathes and operators who each can work 10 hours on a weekend. It takes 18 minutes to perform all needed operations on an ITA, and 12 minutes on an ITS.

The last major operation is drilling the valve stem hole in each wheel. Beau does this himself on another drill press. It takes very little time, but requires a special stepped bit that costs $125 and is worn out after drilling 120 to 130 wheels. This weekend he has one new bit on hand, his old one is worn out, and he's ordered another that won't arrive before next Wednesday.

Beau does some final finishing and packaging himself. The materials for this work cost $12 for an ITA wheel and $ 19.50 for an ITS. His accountant takes $1/wheel depreciation on each of his machines, and figures other overhead at $10/wheel.

Beau would like to put together a practical production schedule for this weekend that would maximize profit for him.

	ITA	ITS
Price	$125.00	$145.00
Casting	$45.00	$50.00
Backfacing	$6.00	$6.00
Drilling	$2.00	$2.50
Lathe	$9.00	$6.00
Stem	$1.00	$1.00
Final	$12.00	$19.50
Marginal Cost	$75.00	$85.00
Contribution Margin	$50.00	$60.00

You should trace this through - each item we included is a direct future cash consequence of making and selling a whee

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Constraints

ITS		$\leq$	100	Demand
ITS	- ITA	$\geq$	0	Prestige
.2 ITS	+ .2 ITA	$\leq$	30	Backface
5 ITS	+ 4 ITA	$\leq$	600	Holes
.2 ITS	+ .3 ITA	$\leq$	30	Lathe
ITS	+ ITA	$\leq$	125	Stem

Dimensionally, this is:

$$ITS \leq ITS \qquad \text{Demand}$$

$$ITS - ITA \geq ITS - ITA \qquad \text{Prestige}$$

$$\left(\frac{\text{hours}}{\text{per ITS}}\right) \times ITS + \left(\frac{\text{hours}}{\text{per ITA}}\right) \times ITA \leq hours \qquad \text{Backface}$$

$$\left(\frac{\text{holes}}{\text{Per ITS}}\right) \times ITS + \left(\frac{\text{holes}}{\text{Per ITA}}\right) \times ITA \leq holes \qquad \text{Holes}$$

$$\left(\frac{\text{hours}}{\text{per ITS}}\right) \times ITS + \left(\frac{\text{hours}}{\text{per ITA}}\right) \times ITA \leq hours \qquad \text{Lathe}$$

$$\left(\frac{\text{holes}}{\text{Per ITS}}\right) \times ITS + \left(\frac{\text{holes}}{\text{Per ITA}}\right) \times ITA \leq holes \qquad \text{Stem}$$

which resolves to

$$ITS \leq ITS \qquad \text{Demand}$$

$$\text{ITS} - ITA \geq ITS - ITA \qquad \text{Prestige}$$

$$hours \leq hours \qquad \text{Backface}$$

$$holes \leq holes \qquad \text{Holes}$$

$$hours \leq hours \qquad \text{Lathe}$$

$$holes \leq holes \qquad \text{Stem}$$

which isn't all that exciting, but it is at least clear that we don't have constraints that are obvious nonsense.

Maximize Z =	60 ITS	+50 ITA				
subject to:	ITS		$\le$	100	ITS	Demand
	ITS	- ITA	$\ge$	0	*"points"*	Prestige
	.2 ITS	+ .2 ITA	$\le$	30	*hours*	Backface
	5 ITS	+ 4 ITA	$\le$	600	*holes*	Holes
	.2 ITS	+ .3 ITA	$\le$	30	*hours*	Lathe
	ITS	+ ITA	$\le$	125	*stemholes*	Stem

To find Beau a usable solution, our first responsibility is to see to it that any proposed solution is ***feasible***.

- A feasible solution is one that does not violate any constraints, <u>including</u> <u>non negativity</u>.

In this case, a feasible solution is a number of ITA and ITS wheels that

- includes either some or none of each product (Beau won't make wheels of antimatter!)
- includes no more than 100 ITS
- has at least as many ITS as ITA
- requires no more than 30 hours of milling machine time to backface the wheels
- requires the drill press operator to drill no more than 600 mounting holes in wheels
- needs no more than 30 lathe hours for truing and surface finishing wheels
- has Beau drilling valve stem holes in no more than a total of 125 wheels.

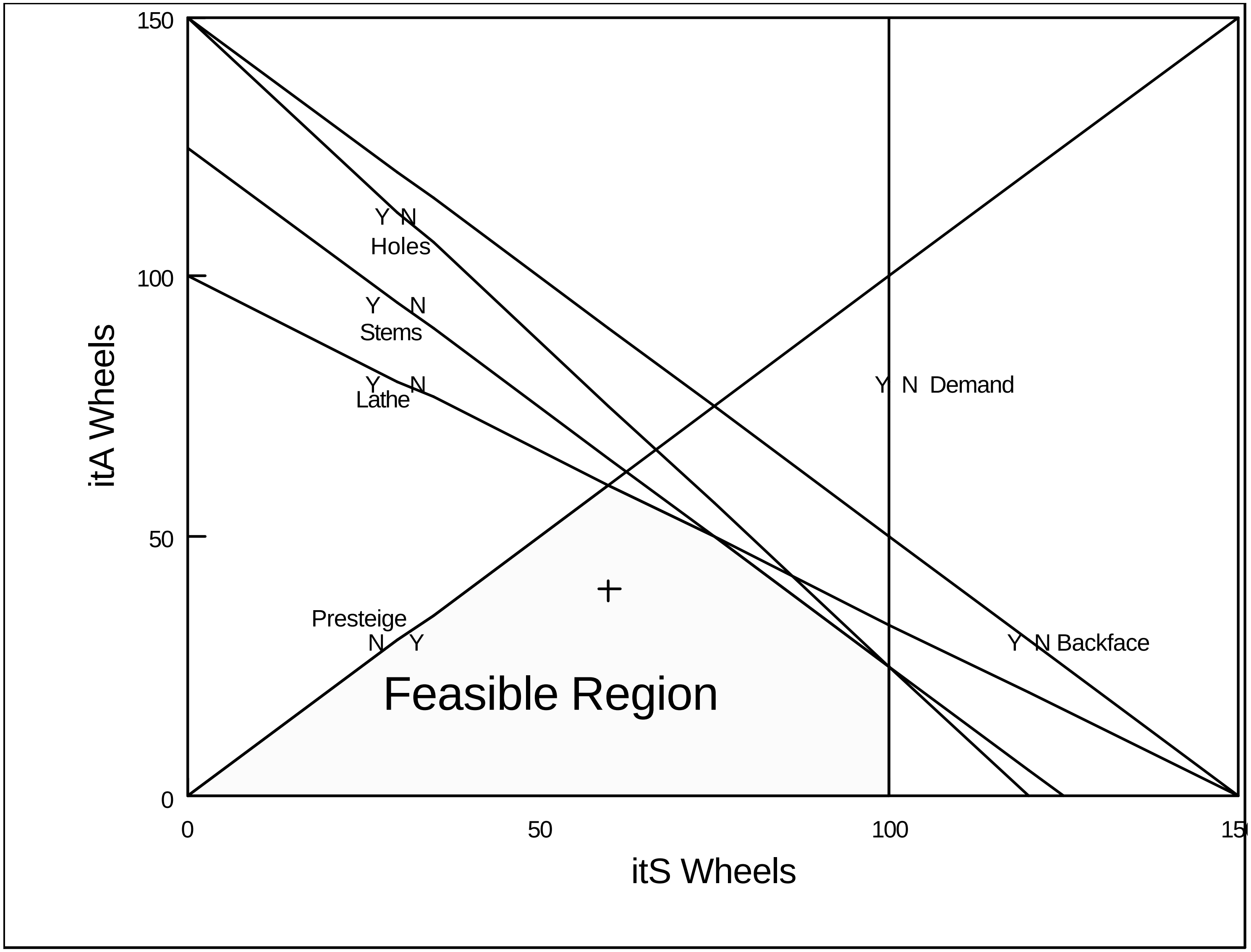
Certainty Assumption

If you're sharp, you have already noticed that we are violating one of the assumptions of LP. Since the special stepped drill bit can drill 120 to 130 holes, but we split the middle and said 125, we are violating the ***certainty*** assumption. The possible risk here is that either at the end of the weekend he finds he could have made more wheels (with a different mix), or that he is stuck with up to 5 wheels that still need valve stem holes drilled. A more conservative approach would have used 120 on the RHS, but given up some profit potential. Later, we will see how to evaluate this. For now, let's just note that we sometimes knowingly violate this assumption, but should always examine the possible consequences.

Feasible solutions

In effect, our inequalities (including the implied $ITA \geq 0$ and $ITS \geq 0$) define a "space", which we call the ***feasible solution space***. We must select a point in that space for Beau, and to do that we'll have to know what points are members of that space. We're going to do that graphically. The graphical representation will help us to understand what LP does and how it works.

Now let's look at Beau's graph:



150
100
50
0
0
50
100
150
itA Wheels
itS Wheels
Y N
Holes
Y N
Stems
Y N
Lathe
Y N Demand
Presteige
N Y
Y N Backface
Feasible Region

Beau Jarble's Constraints

I used **Y(es) and N(o)** to show which side of each constraint line was **infeasable.** The open space at the bottom (labeled Feasible Region) contains all combinations of ITA and ITS wheels Beau can make this weekend. Any point inside this space, or on a boundary, represents a combination that will not violate any constraint. Any point that is not in this region violates at least one constraint.

Beau's problem presents us with some offbeat situations that we sometimes see. First, we have a ***redundant constraint***. Looking at the graph, we can see that the valve stem drilling constraint ***dominates*** the backface constraint. The backface constraint lies entirely outside the feasible region that we would have if the special valve stem drill were the only other constraint. In the olden days, when college students walked barefoot to class through raging blizzards, we suggested that people try to spot and eliminate such constraints before solving LP problems. Now that we have high speed electronic confusers to do the drudge work, we leave them in. If Beau's valve stem drill turned out to be the special titanium version that can drill 1000 holes, we can conceive of the possibility that backfacing might become a binding constraint.

Degeneracy

The other offbeat situation that we have is that this turns out to be a ***degener-ate*** problem. This says nothing about Beau's morals; that's a separate issue. Degeneracy can be a more complex issue when you have more than 2 activity variables, but it is a simple one here. In a 2 dimensional space, the intersection of 2 lines defines a point. But here, we have 3 constraint lines intersecting at ITA = 25, ITS = 100. This is one more line than is needed to define a point in 2 dimensions.

At one time, the mathematicians could not prove that the Simplex Method (we'll see it later) would always find a solution to a degenerate problem. Even today, when someone designs a computer program to solve LP problems with the Simplex Method, he or she must take special precautions to deal with degeneracy.

On a more practical level, after we solve an LP problem, we usually have some interest in ***sensitivity analysis***[1] Sensitivity is often the most important reason for solving an LP problem, and sometimes is even the whole reason.

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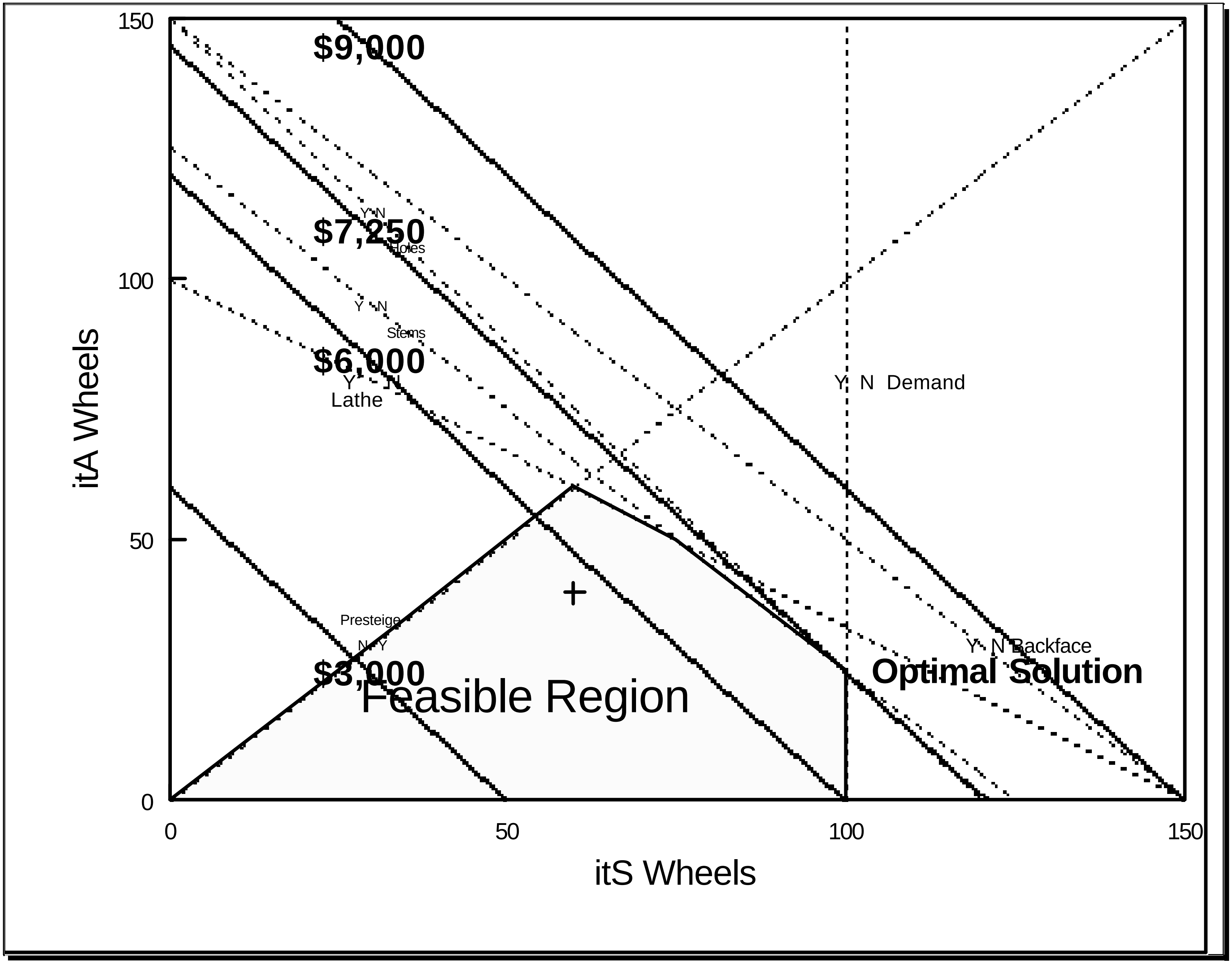
[1]sometimes called ***post-optimality analysis***.

Corner Principle

You will note that we found Beau's best product mix at a (more than usually complicated) intersection of constraints. We call this a ***corner point*** solution. The great breakthrough that made LP a practical problem solving tool instead of just an interesting mathematical structure was the proof of a theorem that permits us to focus on just the corner points. When you demystify the theorem, the essence of what it says is that

- **In a feasible linear programming problem, there is no solution superior to the best corner point solution.**

In Beau's feasible region, there are only 5 corner points. What this theorem says is that one of them *must* be the optimal solution. This is the cornerstone on which the Simplex method, which permits us to solve problems with many variables, is built.



$9,000
Y N
$7,250
Holes
Y N
Stems
$6,000
Y N
Lathe
Y N Demand
Presteige
N Y
$3,000
Feasible Region
Y N Backface
Optimal Solution
150
100
50
0
0
50
100
150
itA Wheels
itS Wheels

Excel Output for Beau Jarble

ItA	ItS			
25	100			
50	60	7,250	Contribution	
0	1	100	Demand	100
0.2	0.2	25	Backface	30
4	5	600	Holes	600
0.3	0.2	27.5	Lathe	30
1	1	125	Stems	125
-1	1	75	Prestige	0
1	0	25	NNItA	0
0	1	100	NNItS	0

Microsoft Excel 5.0 Answer Report

Target Cell (Max)

Cell	Name	Original Value	Final Value
$C$3	Contribution	0	7,250

Adjustable Cells

Cell	Name	Original Value	Final Value
$A$2	ItA	0	25
$B$2	ItS	0	100

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$C$5	Demand	100	$C$5<=$E$5	Binding	0
$C$6	Backface	25	$C$6<=$E$6	Not Binding	5
$C$7	Holes	600	$C$7<=$E$7	Binding	0
$C$8	Lathe	27.5	$C$8<=$E$8	Not Binding	2.5
$C$9	Stems	125	$C$9<=$E$9	Binding	0
$C$10	Prestige	75	$C$10>=$E$10	Not Binding	75
$C$11	NNItA	25	$C$11>=$E$11	Not Binding	25
$C$12	NNItS	100	$C$12>=$E$12	Not Binding	100

Microsoft Excel 5.0 Sensitivity Report

Changing Cells

	Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
	$A$2	ItA	25	0	50	10	2
	$B$2	ItS		0	60	2.5	10
Constraints			100				
	Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
	$C$5	Demand	100	0	100	1E+30	0
	$C$6	Backface	25	0	30	1E+30	5
	$C$7	Holes	600	10	600	0	25
	$C$8	Lathe	27.5	0	30	1E+30	2.5
	$C$9	Stems	125	10	125	3.57	0
	$C$10	Prestige	75	0	0	75	1E+30
	$C$11	NNItA	25	0	0	25	1E+30
	$C$12	NNItS	100	0	0	100	1E+30