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A model of commodity differentiation with indivisibilities and production

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Abstract This paper presents an existence theorem in a general equilibrium model of a production economy with commodity differentiation and indivisibilities. The model is motivated by the existence of markets with indivisible commodities, such as the markets for automobiles and computers. As is standard in the literature, the space of commodity characteristics is described by a compact metric space and a commodity vector is described by an integer-valued Borel measure on the space of commodity characteristics. An atomless measure space of producers and consumers is assumed to overcome the problem of nonconvexity of the production and consumption sets induced by indivisibilities.

Keywords: Existence of equilibrium, commodity differentiation, indivisibilities, production.

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1 Introduction

There is a well established literature on the existence of equilibrium in general equilibrium models with commodity differentiation. A differentiated commodity is viewed as a collection of characteristics, thus there is potentially a continuum of commodities[1]. For instance, Mas-Colell (1975) and Jones (1983) prove the existence of equilibrium in exchange economies with a continuum of consumers and a continuum of commodities[2]. In this paper, we prove the existence of an equilibrium in a general equilibrium model of commodity differentiation with indivisibilities and production.

The literature on general equilibrium models with commodity differentiation can be organized in terms of (i) the availability of production and (ii) the existence of indivisibilities. For instance, Jones (1983) and Anderson and Zame (1998) present models of commodity differentiation with no production and fully divisible commodities. The models in Jones (1984), Zame (1987), and Podczeck (1998) have production and fully divisible commodities. The models in Mas-Colell (1975) and Allen (2005) have no production but the differentiated commodities exhibit indivisibilities. This means the differentiated commodities are only available in whole numbers. Hence, with the exception of Fradera (1986), there are no existence results for models of commodity differentiation with production and indivisibilities.

The significance of indivisibilities in production and consumption can be illustrated by the classic example of a differentiated commodity, namely the automobile. An automobile is made up of several characteristics, such as color, shape, engine power, and transmission type. Rosen (1974) explains his indivisibility assumption by pointing out that two six-foot automobiles are not equivalent to one twelve-foot automobile, because they cannot be driven at the same time. The same is true for other commodities such as houses, computers, and other electronics[3]. Rosen (1974) and Mas-Colell (1975) argue the non-convexity of consumption sets (indivisibilities) is an essential aspect of any model of commodity differentiation.

On the other hand, Jones (1983) points out that models of commodity differentiation with convex consumption sets (fully divisible commodities) have made important contributions and the connections to standard economic models, such as those presented in Debreu (1959) and Hildenbrand (1974), are more easily seen. In addition, working with convex consumption sets allows the existence of equilibria to be guaranteed with only finitely many consumers and producers. It seems that the best way to decide whether convex or non-convex consumption sets are appropriate is to consider the particular differentiated commodity being modeled. For some differentiated commodities, such as automobiles, the indivisibility assumption is crucial for the model to make any sense. For other commodities, such as milk, convex consumption sets may be more acceptable.

[1] For additional discussion on differentiated commodities, see Lancaster (1966). Hart (1979) and Suzuki (2000) present models of commodity differentiation and monopolistic competition.

[2] In addition, see Ostroy (1984) for existence of equilibrium in large-square economies.

[3] For an application to insurance policies see Marton (2006).

As mentioned, a shortcoming of this literature is that there has been very little work done on existence theorems for models of commodity differentiation with indivisibilities and production. Mas-Colell (1975) states and proves an existence theorem for an exchange economy. This leads to the sometimes unnatural interpretation of endowing consumers with the differentiated commodity. Fradera (1986) proves an existence theorem with production, but does so with a restricted consumption set in which consumers are only allowed to consume at most one unit of the differentiated commodity. In addition, the model does not allow differentiated commodities to be used as inputs in the production process.

The main contribution of our paper is to prove an existence theorem for a general equilibrium model of commodity differentiation with indivisibilities and production. Unlike Fradera (1986), consumers in our model are allowed to consume multiple units of the differentiated commodities and the differentiated commodities can be used as inputs in the production process. The description of consumers and commodities in our model follows closely Mas-Colell (1975) while the description of the production technology is similar to Jones (1984) except for the indivisibilities. As in Podczeck (1998), we assume a continuum of producers. Hence, our paper can also be interpreted as an existence theorem for a general equilibrium economy with a continuum of consumers and producers and a continuum of commodities, where commodities exhibit indivisibilities[4].

The model presented in this paper can be used to model markets such as the markets for automobiles and computers. This is because it seems natural to think of an automobile or a computer as a bundle of individual characteristics. Moreover, the key components of the model, indivisibilities and production, are both important in these markets. For instance, without indivisibilities, consumers could purchase non-integer quantities of automobiles, which does not seem to make much sense. In addition, without production, consumers would have to be endowed with automobiles, which also does not seem plausible.

The paper is organized as follows. Section II presents the model and assumptions and provides additional discussion of how the model is related to the literature. Section III presents the proof of existence.

2 The Model

2.1 The space of commodity characteristics

There is a given compact metric space K, called the **space of commodity characteristics**. A typical element is denoted by $k \in K$. The Borel σ-algebra of subsets of K is denoted by $\beta(K)$ and a typical Borel subset of K is denoted by $V \in \beta(K)$. Finally, denote by d_K a metric on K. A point $k \in K$ is a complete description of one unit of a commodity, and commodities are close if the characteristics of their units are topologically close in K.

[4] For an introduction to general equilibrium models with an infinite number of commodities see Aliprantis, Brown, and Burkinshaw (1990).

There is a **homogeneous commodity**, denoted by h, which is not subject to differentiation. The commodity characteristics space K is made up of two components, the compact space of **differentiated** commodity characteristics, denoted by K^D, and the homogeneous commodity. Therefore, we have

$$K = K^D \cup \{h\}. \tag{1}$$

The homogeneous commodity h is assumed to be non-differentiated, and hence $h \notin K^D$. The inclusion of one homogeneous commodity is essential for the existence of equilibrium. This commodity can generally be thought of as numeraire or money.

2.2 Consumers

Denote by $M(K)$ the space of bounded signed Borel measures on K. For any signed measure $m \in M(K)$, denote by m^+ and m^- the positive and negative parts of m, respectively. Denote by $\|\cdot\|_V$ the variation norm on $M(K)$. Formally, for any $m \in M(K)$, we have

$$\|m\|_V = |m|(K) = m^+(K) + m^-(K). \tag{2}$$

The space $M(K)$ with the variation norm is the dual of the space $C(K)$, the space of real-valued continuous functions on K, with the supremum norm.[5]

The space $M(K)$ is endowed with the weak* topology. The weak* topology on $M(K)$ is the topology of pointwise convergence on the continuous functions in $C(K)$, namely

$$\mu_n \to \mu \text{ iff } \int g\, d\mu_n \to \int g\, d\mu \text{ for every } g \in C(K).$$

The weak* topology on $M(K)$ is separable and norm-bounded subsets of $M(K)$ are metrizable. In particular, the α-ball of $M(K)$, namely

$$M_\alpha(K) = \{m \in M(K) : |m|(K) \leq \alpha\} \tag{3}$$

is weak* compact and metrizable. Moreover, a subset $A \subset M(K)$ is weak* compact if and only if it is weak* closed and norm-bounded (Aliprantis and Border (1999), p. 250).

2.2.1 Commodity bundles

A **commodity bundle** is a measure $m \in M(K)$ such that the restriction to K^D, denoted by $m \mid K^D$, is integer valued. Denote by Ω the set of all commodity bundles, namely

$$\Omega = \left\{m \in M(K) : m \mid K^D \text{ is integer valued}\right\}. \tag{4}$$

Denote by Ω_+ the set of non-negative commodity bundles.

[5]The supremum norm is given by $\|g\|_\infty = \sup\{|g(k)| : k \in K\}$. Much of the background material on infinite dimensional analysis can be found in Aliprantis and Border (1999) and Hildenbrand (1974).

2.2.2 Consumption bundles

An **individual consumption bundle** $m \in \Omega_+$ is a non-negative, bounded Borel measure on K such that the restriction to K^D is integer valued. Hence, an individual consumption bundle is a purely atomic measure with a finite number of atoms.

The interpretation is that if a consumer chooses m as his consumption bundle, $m(V)$ is the total quantity of all commodities with characteristics in $V \in \beta(K)$.

2.2.3 The consumption Set

Fix a positive integer $\alpha^* \in \mathbb{N} = \{1, 2, \cdots\}$.[6] The **consumption set**, denoted $X \subset \Omega_+ \subset M(K)$, is the set of all individual consumption bundles m such that $m(K^D) \leq \alpha^*$, namely

$$X = \left\{m \in \Omega_+ : m\left(K^D\right) \leq \alpha^*\right\}. \tag{5}$$

The set of all commodity bundles $\Omega \subset M(K)$ and the consumption set $X \subset M(K)$ are endowed with the weak* topology. Any reference to convergence of measures is with respect to the weak* topology.

Let d_M be a metric for the weak* topology on the α^*-ball of $M\left(K^D\right)$. Then, the metric d_X on X is given by

$$d_X(m, m') = d_M(m \,|\, K^D, m' \,|\, K^D) + \|m(h) - m'(h)\|$$

where $\|\cdot\|$ denotes the Euclidean norm. Hence, m' is close to m if $m'(h)$ is close to $m(h)$ and $m' \,|\, K^D$ is obtained from $m \,|\, K^D$ by a small perturbation of the characteristics of the commodities in $m \,|\, K^D$.

2.2.4 Preferences

Denote by $b \in X$ the commodity bundle with one unit of the homogeneous commodity. Formally, for any $V \in \beta(K)$, we have $b(V) = 1$ if $h \in V$ and $b(V) = 0$ otherwise. Denote by δ_k the Dirac measure at $k \in K$.

Assumption (P) A **preference relation** is a complete preorder $\succsim \subset X \times X$ with the properties[7]:

P1. $\succsim$ is closed.

P2. if $m' \geq m$ and $m'(h) > m(h)$, then $m' \succ m$.

P3. if $m'(h) > 0$ and $m(h) = 0$, then $m' \succ m$.

[6] The integer α^* should be interpreted as a very large integer, which is the intended interpretation in Mas-Colell (1975). In Fradera (1986), α^* is set equal to one.

[7] For any μ, $\mu' \in M(K)$, (i) $\mu \leq \mu'$ means $\mu(V) \leq \mu'(V)$ for every $V \in \beta(K)$, (ii) $\mu < \mu'$ means $\mu(V) < \mu'(V)$ for every $V \in \beta(K)$, and (iii) $\mu \succ \mu'$ iff $\mu \succsim \mu'$ but not $\mu' \succsim \mu$.

P4. There exists a real $\gamma > 0$ such that for all $k \in K^D$ and $m \in X$ with $m(k) > 0$, we have

$$m - \delta_k + \gamma b \succ m.$$

Assumptions P1 and P2 are standard and imply preferences are continuous and strictly monotone on the homogeneous commodity. Assumptions P3 implies the homogeneous commodity is a "necessity". This assumption is needed because of the indivisibilities (Mas-Colell, 1975). Assumption P4 says the homogeneous commodity can be substituted for any differentiated commodity at a bounded rate (Jones, 1984). This assumption is needed to prove equilibrium prices are bounded.

The assumptions on preferences follow more closely the assumptions in Mas-Colell (1975) than the assumptions in Jones (1984) or Podczeck (1998)[8]. In particular, assumptions P1-P3 are the same as in Mas-Colell (1975) while assumption P4 is similar to Jones (1984). This is because our model and the model in Mas-Colell (1975) deal with indivisibilities, while the others assume fully divisible commodities. The assumptions on preferences differ from the assumptions in Mas-Colell (1975) because we assume all consumers are endowed with a positive quantity of the homogeneous commodity while Mas-Colell (1975) assumes all commodities are available in the aggregate.

Let $\zeta(X \times X)$ be the space of nonempty, closed subsets of $X \times X$. $\zeta(X \times X)$ with the closed convergence topology is a separable, compact metric space.[9] Denote by $\mathbb{P}$ the **set of all preference relations**. Endow $\mathbb{P} \subset \zeta(X \times X)$ with the relativized closed convergence topology and denote by d_P a metric for this topology.

2.2.5 Consumer characteristics

We describe consumers in this model with the following assumption.

Assumption (C) There are compact subsets $P \subset \mathbb{P}$ and $E \subset X$ of **consumer characteristics**. A **consumer** is a pair $(\succsim, \omega) \in P \times E$.

Assumption C is standard and it implies that consumers are not too dissimilar. Because we assume constant returns to scale, we do not need to describe consumer ownership shares of production.

In addition, as in Mas-Colell (1975) and Jones (1983), we need to make a uniform assumption on preferences.

Assumption (BP) Assumption P4 holds uniformly on P. Namely, there exists $\gamma > 0$ such that for all $\succsim \in P$, $k \in K^D$ and $m \in X$ with $m(k) > 0$, we

[8] Podczeck (1998) assumes the consumption set is endowed with the bounded weak* topology and preferences are bounded weak* continuous. As discussed in Podczeck (1998), the bounded weak* topology and the weak* topology agree on the consumption set, so bounded weak* continuous preferences is the same as weak* continuous preferences.

[9] See Hildenbrand (1974, Part I, BII, Theorem 2).

have

$$m - \delta_k + \gamma b \succ m.$$

This assumption guarantees the marginal rates of substitution are bounded uniformly. As stated in Jones (1984), this is essentially a boundary condition that guarantees prices can be appropriately assigned to commodities that are not available in the aggregate. Our assumption is different from the uniform assumptions in Mas-Colell (1975) and Jones (1983). For instance, Mas-Colell (1975) assumes there exists some quantity of the homogenous commodity such that all consumers would trade "similar" differentiated commodities provided they are compensated with the given quantity of the homogenous commodity. We need assumption BP because we do not assume all commodities are available in the aggregate.

2.3 Producers

The **production sector** of the economy is described by a continuum of producers $F = [0, 1]$ and a production technology $Y \subset \Omega$. The interpretation is that all the producers have access to the same production technology[10]. Denote by $\beta(F)$ the collection of Borel subsets of F and let λ denote the Lebesgue measure. Hence, $(F, \beta(F), \lambda)$ is an atomless measure space. Y is endowed with the relative topology from $M(K)$.

A **production vector** $y \in Y$ is a commodity vector, where we follow the standard convention and represent inputs with negative entries and outputs with positive entries.

2.3.1 The production set

For a collection of commodities $\{k_1, \cdots, k_n\} \subset K$, denote by $\mathrm{LS}(k_1, \cdots, k_n)$ the **linear subspace** of $M(K)$ generated by the collection $\{\delta_{k_1}, \cdots, \delta_{k_n}\}$, where δ_k denotes the Dirac measure at $k \in K$.

Assumption (F) The production set Y has the following properties:

F1. Standard:

(a) Y is closed.

(b) $0 \in Y$ and $Y - \Omega_+ \subset Y$.

F2. CRS: $y \in Y$ implies $\alpha y \in Y$ for all $\alpha \in \mathbb{N}$.

[10] We have introduced the production sector as a *representation* $\varphi : F \to \{Y\}$ while we introduced the consumption sector as a probability measure over a space of consumer characteristics. Alternatively, and to maintain the parallel with the consumption sector, one can interpret the production sector as a probability measure over the collection $\{Y\}$, which contains one element only. The formulation of the production sector can be extended to a probability measure over the collection $\{Y_1, \cdots, Y_L\}$ without any substantive change in the analysis or the results but with a considerable cost in terms of notation.

F3. Substitutability:

(a) For all $\sigma > 0$ and for all sequences $y^n \in Y$ and $k^n, \hat{k}^n \in K$ such that $k^n \to k$, $\hat{k}^n \to k$, $y^n \to y$ and $y^n(k^n) < 0$, there is an $n \in \mathbb{N}$ such that

$$y^n + \delta_{k^n} - \delta_{\hat{k}^n} - \sigma b \in Y.$$

(b) For all sequences $y^n \in Y$, $k^n \in K$, and $q^n \in \mathbb{R}_+$ such that $y^n \to y$, $k^n \to k$, $q^n \to \infty$, and $y^n(k^n) < 0$, there exists an $n \in \mathbb{N}$ such that

$$y^n + \delta_{k^n} - q^n b \in Y.$$

F4. Approximation: There exists a countable set $\{k_1, k_2, \cdots\}$ in K such that $Y \cap (\cup_{n \in \mathbb{N}} \mathrm{LS}\{k_1, \cdots, k_n\})$ is dense in Y.

Assumption F1 is standard and implies the possibility of inaction and free disposal[11]. Assumption F2 of constant returns to scale technology is not as restrictive as it may seem. McKenzie (1959) pointed out that it is possible to obtain this property through an appropriate introduction of an "entrepreneurial factor". The substitutability assumptions are necessary to ensure that the equilibrium prices are continuous. Assumption F3(a) says that similar inputs can be substituted for each other at nearly the same rate. Assumption F3(b) says that the homogeneous factor can always be substituted for any other input at a bounded rate. These assumptions differ from Jones (1984) because of the indivisibilities. Assumption F4 says that any production plan for the firm can be approximated in Y using only the commodities in $\{k_1, k_2, \cdots\}$ as inputs and outputs. As in Jones (1984), we will use this assumption in the construction of the sequence of approximating economies.

At this point, some discussion of the differences between our production sector and those in other papers in the literature is appropriate. In Fradera (1986), production is restricted so that the only input is the homogeneous commodity, the differentiated commodities are the only outputs, and there is no joint production. The production set described here is more general. Mas-Colell (1975) and Jones (1984) endow the commodity space with the weak* topology. An alternative, presented in Podczeck (1998), is to endow the commodity space with the bounded weak* topology. Podczeck (1998) shows how the uniform properness assumption made on production under the bounded weak* topology is implied by the substitutability assumptions made by Jones (1984) under the weak* topology. The weak star topology is used in this paper to accommodate the indivisibilities present in this model and Mas-Colell (1975).

2.3.2 The aggregate production set

For every $V \in \beta(K)$, the function that maps $m \in \Omega$ to $m(V) \in \mathbb{R}$ is measurable (Fact 1 of the appendix). Denote by μ a probability measure on $(\Omega, \beta(\Omega))$ and

[11] In addition, we will impose below the properties of irreversibility and no free production on the aggregate production set.

denote by $i : \Omega \to \Omega$ the identity map. Then, if $\int_\Omega |m|(K)\, d\mu(m) < \infty$, we denote by $\int_\Omega i\, d\mu$ the element of $M(K)$ defined by

$$\begin{aligned}\left(\int_\Omega i\, d\mu\right)(V) &= \int_\Omega m(V)\, d\mu(m) \\ &= \int_\Omega m^+(V)\, d\mu(m) - \int_\Omega m^-(V)\, d\mu(m)\ , \quad V \in \beta(K)\,.\end{aligned}$$

In this situation, we say the identity map is μ-integrable and the integral $\int_\Omega i\, d\mu$ is interpreted as the aggregate commodity distribution.

For any subset $A \subset M(K)$, denote by $M_+^1(A)$ the set of probability measures on $(A, \beta(A))$, where A is endowed with the topology of $M(K)$. The **aggregate production set** of the economy is given by

$$\mathbb{Y} = \left\{ y \in M(K) : y = \int_Y i\, d\sigma\ ,\ \ \sigma \in M_+^1(Y) \right\}. \tag{6}$$

Denote by $\zeta(K)$ the space of compact subsets of K which include the homogeneous commodity, namely

$$\zeta(K) = \{J \subset K : J \text{ is closed and } \{h\} \subset J\}. \tag{7}$$

Denote by Ω_J the restriction of Ω to the elements of $J \in \zeta(K)$, namely

$$\Omega_J = \{m \in \Omega : m\,|\,(K\backslash J) = 0\}. \tag{8}$$

In addition, denote by Y_J the restriction of Y to the elements of $J \in \zeta(K)$ and denote by $\mathbb{Y}_J$ the associated aggregate production set, namely

$$Y_J = Y \cap \Omega_J$$

and

$$\mathbb{Y}_J = \left\{ y \in M(K) : y = \int_{Y_J} i\, d\sigma\ ,\ \ \sigma \in M_+^1(Y_J) \right\}.$$

For any bounded subset $\tilde{E} \subset \Omega_+$ of endowments, denote by $\tilde{E}_J = \tilde{E} \cap \Omega_J$ the restriction of $\tilde{E}$ to the commodities in $J \in \zeta(K)$. Denote by $\Lambda(\mathbb{Y}_J, \tilde{E}_J)$ the set of possible aggregate production vectors when the economy is restricted to the commodities in $J \in \zeta(K)$, namely

$$\Lambda(\mathbb{Y}_J, \tilde{E}_J) = \mathbb{Y}_J \cap \left\{ y \in M(K) : y + \int_{\tilde{E}_J} i\, d\mu \geq 0\ ,\ \ \mu \in M_+^1(\tilde{E}_J) \right\}. \tag{9}$$

For any $\theta \in \mathbb{N}$ and $J \in \zeta(K)$, consider the set of norm-bounded individual production vectors $Y_J^\theta = \{y \in Y_J : |y|(K) \leq \theta\}$ and denote by $\mathbb{Y}_J^\theta$ the associated aggregate production set, namely

$$\mathbb{Y}_J^\theta = \left\{ y \in \mathbb{Y}_J : y = \int_{Y_J^\theta} i\, d\sigma\ ,\ \ \sigma \in M_+^1\left(Y_J^\theta\right) \right\}.$$

Assumption (BF) The aggregate production set $\mathbb{Y}$ has the following properties:

(a) Standard: $\mathbb{Y} \cap (-\mathbb{Y}) = \{0\}$ and $\mathbb{Y} \cap M_+(K) = \{0\}$.

(b) Boundedness: For every bounded subset $\tilde{E} \subset \Omega_+$, there exists a $\theta \in \mathbb{N}$ such that, for all $J \in \zeta(K)$, we have

$$\Lambda(\mathbb{Y}_J, \tilde{E}_J) \subset \mathbb{Y}_J^{\theta}.$$

Assumption BF(a) contains the standard irreversibility and no free production assumptions. Assumption BF(b) clearly implies $\Lambda(\mathbb{Y}, \tilde{E}) \subset \mathbb{Y}^{\theta}$ is bounded. In other words, if aggregate resources are bounded, then the possible aggregate production vectors are bounded as well. In addition, assumption BF(b) implies there is a uniform bound on the scale of "elementary" production activities.[12] In particular, if an aggregate production vector is bounded, then it can be produced with bounded individual production vectors, and this boundedness assumption continues to hold if we restrict attention to production vectors that involve only a subset of commodities. The production sets in Fradera (1986) and Podczeck (1992) satisfy this assumption. In addition, Podczeck (1998) uses a similar condition to show the aggregate production set is closed.

2.4 Prices

Recall $C(K)$ denotes the space of real-valued continuous functions on K with the supremum norm. A **price system** p is an element of $C^+(K)$, where

$$C^+(K) = \{p \in C(K) : p \geq 0 \text{ and } p(h) > 0\}. \tag{10}$$

In other words, it is imposed as a condition that similar commodities have similar prices.

For any $m \in M(K)$ and $p \in C^+(K)$, the value of m at prices p is denoted by $p \cdot m$ and can be calculated as

$$p \cdot m = \int_K p \, dm. \tag{11}$$

2.5 The Economy

Endow $P \times E$ with the product topology, so that $\beta(P \times E)$ is the corresponding Borel σ-algebra. Denote by v a probability measure on $(P \times E, \beta(P \times E))$. Denote by v_E the probability measure v restricted to domain $(E, \beta(E))$.

[12]For instance, if the production set Y is spanned by a collection $A \subset \Omega$ of elementary activities, as in Podczeck (1992), then assumption (BF) implies A is norm-bounded.

2.5.1 Description

An **economy** $\mathcal{E}$ is a list $[v,(F,Y)]$ where v is a probability measure on $(P \times E, \beta(P \times E))$ such that:

(i) $\int_E m(h)\, dv_E(m) < \infty$,

(ii) $\omega(h) > 0$ for all $\omega \in \text{supp}(v_E)$, and

(iii) $\omega\left(K^D\right) < \alpha^*$ for all $\omega \in \text{supp}(v_E)$.

Condition (i) says that the mean amount of the homogeneous commodity is finite. Condition (ii) says that all consumers are endowed with a positive quantity of the homogeneous commodity. Finally, condition (iii) says that all consumers are endowed with a total quantity of the differentiated commodities that is strictly less than the total quantity they could possibly consume[13]. As above, $\int_E i\, dv_E$ is the aggregate commodity distribution corresponding to v_E and is interpreted as the aggregate endowment.

2.5.2 Equilibrium

Denote by $B(p,\omega) = \{m \in X : p \cdot m \leq p \cdot \omega\}$ the **budget set** of a consumer. For every $p \in C^+(K)$, let

$$H_C(K,p) = \{(\succsim, \omega, m) \in P \times E \times X : m \text{ maximizes } \succsim \text{ in } B(p,\omega)\}. \quad (12)$$

$H_C(K,p)$ is the set of all bundles of preference relations, endowments, and consumption measures such that the consumption measure is affordable and maximizes preferences.

In addition, denote by $H_F(K,p)$ the set of production vectors that are profit maximizing at $p \in C^+(K)$, namely

$$H_F(K,p) = \{y \in Y : p \cdot y \geq p \cdot \hat{y} \text{ for all } \hat{y} \in Y\}. \quad (13)$$

Denote by τ a probability measure on $(P \times E \times X, \beta(P \times E \times X))$. In addition, denote by $\tau_{P\times E}$ the probability measure τ restricted to domain $(P \times E, \beta(P \times E))$ and denote by τ_X the probability measure τ restricted to domain $(X, \beta(X))$.

Definition 1 *An* ***equilibrium*** *of economy $\mathcal{E}$ consists of a probability measure τ on $(P \times E \times X, \beta(P \times E \times X))$, a probability measure σ on $(Y, \beta(Y))$, and a price system $p \in C^+(K)$ such that $\tau_{P\times E} = v$ and*

(i) $\tau[H_C(K,p)] = 1$,

(ii) $\sigma[H_F(K,p)] = 1$, *and*

[13] As discussed previously, the integer α^* should be interpreted as a very large integer. Therefore, condition (iii) is mainly a technical assumption.

(iii) $\int_X i \, d\tau_X = \int_E i \, d\tau_E + \int_Y i \, d\sigma$.

Conditions (i) and (ii) represent utility maximization and profit maximization, respectively. Condition (iii) implies aggregate consumption is equal to aggregate supply.

We now state the main result of the paper.

Theorem 1 *For any economy $\mathcal{E}$ there is an equilibrium.*

3 Proof of Existence

The strategy used to prove this existence theorem is similar to the strategy used in Mas-Colell (1975) and Jones (1984). In particular, we construct (i) a sequence (K_n) of subsets of K, where $K_n = \{k_1, \cdots, k_{m_n}\}$ and $K_n \to K$, and a (ii) a sequence of economies $(\mathcal{E}^n)$, where the commodity space in economy $\mathcal{E}^n$ is restricted to $K_n \subset K$. Then, we use an existence theorem from Hildenbrand (1974) to show the economy $\mathcal{E}^n$ has a equilibrium. Finally, compactness is used to show a subsequence of equilibria converges and the limit is an equilibrium of economy $\mathcal{E}$.

3.1 Preliminaries

Define the measure space of consumers as $(I, \beta(I), \lambda)$, where $I = [0,1]$, $\beta(I)$ represents the set of Borel subsets of I, and λ denotes the Lebesgue measure.

3.1.1 Representations and assignments

A **representation** for an economy $\mathcal{E}$ is a measurable function $e : I \to P \times E$ such that $v = \lambda \circ e^{-1}$. Since $P \times E$ is a separable and complete metric space, any distribution v has a representation.[14] Sometimes it will be convenient to write $(\succsim^t, \omega(t))$ for $e(t)$. The measurability of the representation e implies (i) for every $V \in \beta(K)$, the function that maps $t \in I$ to $\omega(t)(V)$ is measurable and (ii) for every m, $m' \in X$, $\{t \in I : m \succsim_t m'\}$ is a measurable subset of I.

A **consumption assignment** $a : I \to X$ is a function such that for every $V \in \beta(K)$, the function that maps $t \in I$ to $a(t)(V)$ is measurable. Given a consumption assignment with $\int_I a(t)(K) \, d\lambda(t) < \infty$, the integral $\int_I a \, d\lambda$ denotes the element of $M(K)$ defined as $\left(\int_I a \, d\lambda\right)(V) = \int_I a(t)(V) \, d\lambda(t)$, for all $V \in \beta(K)$ (Fact 2 of the appendix). The integral $\int_I a \, d\lambda$ is interpreted as the aggregate consumption distribution.

A **production assignment** $f : F \to Y$ is a function such that for every $V \in \beta(K)$, the functions that map $\iota \in F$ to $f(\iota)^+(V)$ and $f(\iota)^-(V)$ are measurable. Given a production assignment with $\int_F |f(\iota)|(K) \, d\lambda(\iota) < \infty$, the integral $\int_F f d\lambda$ denotes the element of $M(K)$ defined as $\left(\int_F f d\lambda\right)(V) =$

[14] See Hildenbrand (1974, p. 50).

$\int_F f(\iota)^+(V)\,d\lambda(\iota) - \int_F f(\iota)^-(V)\,d\lambda(\iota)$, for all $V \in \beta(K)$ (Fact 2 of the appendix). In this situation, we say f is λ-integrable and the integral $\int_F f d\lambda$ is interpreted as the aggregate production distribution.

An **allocation** for an economy $\mathcal{E}$ is a pair of assignments (a, f) such that $\int_I a\,d\lambda \leq \int_E i\,dv_E + \int_F f\,d\lambda$.

A probability measure τ on $P \times E \times X$ is represented by $e : I \to P \times E$ and $a : I \to X$ if $\tau = \lambda \circ (e, a)^{-1}$. Since $P \times E \times X$ is a separable and complete metric space, any distribution on $P \times E \times X$ has a representation.

3.1.2 Intermediate results

The material presented in this section, and a number of the intermediate results, follows closely Mas-Colell (1975).

Assume the set of endowments has the form $E = \{m \in X : m(h) \leq \beta\}$ and denote by $C_1^+(K) = \{p \in C^+(K) : p(h) = 1\}$ the space of normalized prices.

Recall $\zeta(K)$ denotes the space of compact subsets of K which include the homogeneous commodity and Ω_J denotes the restriction of Ω to the elements of $J \in \zeta(K)$. The space $\zeta(K)$ endowed with the closed convergence topology is a compact, metrizable space.

The other elements of the model, restricted to the commodities in $J \in \zeta(K)$, are described as follows:

1. Consumers

   (a) Consumption set
   $$X_J = X \cap \Omega_J$$

   (b) Preferences
   $$\begin{aligned} \mathbb{P}_J &= \{\succsim \cap X_J \times X_J : \succsim \in \mathbb{P}\} \\ P_J &= \{\succsim \cap X_J \times X_J : \succsim \in P\} \end{aligned}$$

   Let
   $$\hat{\mathbb{P}} = \bigcup_{J \in \zeta(K)} \mathbb{P}_J \quad \text{and} \quad \hat{P} = \bigcup_{J \in \zeta(K)} P_J$$

   where $\hat{\mathbb{P}}$ and $\hat{P}$ are endowed with the closed convergence topology.

   (c) Endowments
   $$E_J = E \cap X_J$$

2. Production
   $$Y_J = Y \cap \Omega_J$$

3. Prices
   $$C_1^+(J) = \left\{p \in C_1^+(K) : p(k) = 0 \text{ for all } k \in K \backslash J\right\}$$

Lemma 1 (Mas-Colell) *The set* $\left\{(m, \hat{m}, \succsim) \in X_J \times X_J \times \hat{\mathbb{P}} : m \succsim \hat{m}\right\}$ *is closed.*

Lemma 2 (Mas-Colell) *The function* $\Psi : \zeta(K) \times \mathbb{P} \to \hat{\mathbb{P}}$ *given by*

$$(J, \succsim) \to \succsim \cap X_J \times X_J$$

is continuous.

Lemma 3 (Mas-Colell) $\hat{P}$ *is compact.*

We now introduce the notion of **convergence** used in Mas-Colell (1975) for the pair $J \in \zeta(K)$ and $p \in C_1^+(J)$. For J_n, $J \in \zeta(K)$, $p_n \in C_1^+(J_n)$, and $p \in C_1^+(J)$, we write $(J_n, p_n) \to (J, p)$ if and only if $J_n \to J$ and, for every subsequence (J_{n_i}, p_{n_i}), $k_{n_i} \to k$ and $k_{n_i} \in J_{n_i}$ implies $p_{n_i}(k_{n_i}) \to p(k)$. From now on, when (J, p) is written, it implies $J \in \zeta(K)$ and $p \in C_1^+(J)$.

Lemma 4 (Mas-Colell) *Suppose* $(J_n, p_n) \to (K, p)$ *and* $m_n \in \Omega_{J_n}$, $m_n \to m$. *Then,* $p_n \cdot m_n \to p \cdot m$.

For any $p \in C_1^+(J)$ and $\omega \in E_J$, let $B_J(p, \omega) = \{m \in X_J : p \cdot m \leq p \cdot \omega\}$ denote the budget set restricted to the commodities in $J \in \zeta(K)$ and let

$$H_C(J, p) = \{(\succsim, \omega, m) \in P_J \times E_J \times X_J : m \text{ maximizes } \succsim \text{ in } B_J(p, \omega)\}$$

and

$$H_F(J, p) = \{y \in Y_J : p \cdot y \geq p \cdot \hat{y} \text{ for all } \hat{y} \in Y_J\}.$$

Lemma 5 *Assume there is a countable set* $\{k_1, k_2, \cdots\}$ *in* K *such that* $Y \cap (\cup_{n\in\mathbb{N}} \mathrm{LS}\{k_1, \cdots, k_n\})$ *is dense in* Y. *In addition, suppose* $\{k_1, \cdots, k_n\} \subset J_n$ *and* $(J_n, p_n) \to (K, p)$. *Then,* (i) $\mathrm{Ls}(H_C(J_n, p_n)) \subset H_C(K, p)$ *and* (ii) $\mathrm{Ls}(H_F(J_n, p_n)) \subset H_F(K, p)$.[15]

Proof. (i) This is proved in Lemma 5 in Mas-Colell (1975). (ii) To prove $\mathrm{Ls}(H_F(J_n, p_n)) \subset H_F(K, p)$ it suffices to show that $y_n \in H_F(J_n, p_n)$ and $y_n \to y \in Y$ implies $y \in H_F(K, p)$. For any $\hat{y} \in Y \cap (\cup_{n\in\mathbb{N}} \mathrm{LS}\{k_1, \cdots, k_n\})$, because $\{k_1, \cdots, k_n\} \subset J_n$, we must have $p_n y_n \geq p_n \hat{y}$ for n large enough. Moreover, because $p_n \to p$ and $p_n y_n \to py$ (Lemma 4), we must have $py \geq p\hat{y}$. Finally, because $Y \cap (\cup_{n\in\mathbb{N}} \mathrm{LS}\{k_1, \cdots, k_n\})$ is dense in Y, it follows that $py \geq p\hat{y}$ for all $\hat{y} \in Y$. Therefore, $y \in H_F(K, p)$. ■

Let $J_n \in \zeta(K)$ and $p_n \in C_1^+(J_n)$. The sequence (J_n, p_n) will be called **equicontinuous** if for every $\varepsilon > 0$ there is a $\delta > 0$ such that, for all n, if k, $k' \in J_n$ and $d_K(k, k') \leq \delta$, then $| p_n(k) - p_n(k') | \leq \varepsilon$.

Lemma 6 (Mas-Colell) *Let* $J_n \in \zeta(K)$ *and* $p_n \in C_1^+(J_n)$, $J_n \to K$, *and* $J_n \subset J_{n+1}$ *for all* n. *If the sequence* (J_n, p_n) *is equicontinuous and for some* η, $p_n(k) \leq \eta$ *for all* n *and* $k \in K$, *then there exists a* $p \in C_1^+(K)$ *such that* $(J_{n_i}, p_{n_i}) \to (K, p)$ *for a subsequence* (J_{n_i}, p_{n_i}) *of* (J_n, p_n).

[15] For the definition of Ls, see Hildenbrand (1974, p. 15).

Lemma 7 *The sets $H_C(K,p)$ and $H_F(K,p)$ are measurable.*

Proof. In order to prove the sets are measurable, it suffices to show they are closed. The closedness of $H_C(K,p)$ follows from Lemma 5.

We will now show $H_F(K,p)$ is a closed. A set is closed if and only if it contains all of its limit points. Hence, suppose $y \in M(K)$ is a limit point of $H_F(K,p)$. Denote by N_y a neighborhood base at y. Then, for each $U \in N_y$ we have

$$(U \backslash \{y\}) \cap H_F(K,p) \neq \varnothing.$$

For each $U \in N_y$, take $x_U \in (U \backslash \{y\}) \cap H_F(K,p)$. Then, because $x_U \in H_F(K,p) \subset Y$, $\{x_U\}_{U \in N_y}$ is a net in Y and converges to y. Moreover, because Y is closed, we have $y \in Y$.

Take any $\hat{y} \in Y$. Then, because $x_U \in H_F(K,p)$, we have $p \cdot x_U \geq p \cdot \hat{y}$ for all $U \in N_y$. Finally, because $x_U \to y$, we obtain $p \cdot y \geq p \cdot \hat{y}$ and therefore $y \in H_F(K,p)$. Hence, $H_F(K,p)$ is closed. ■

3.2 Sequence of approximating economies

We now construct a sequence $\mathcal{E}^n$ of approximate economies based on a particular sequence $K_n \in \zeta(K)$ such that $K_n \to K$.

3.2.1 Construction of economy $\mathcal{E}^n$

First, we will describe the particular sequence K_n. Then, we will describe the consumers and producers in approximate economy $\mathcal{E}^n$.

Construction of K_n

The space K of commodity characteristics is a complete separable metric space, and hence there exists a countable set $\{k_1, k_2, \cdots\}$ dense in K such that (i) $k_1 = h$ and (ii) $Y \cap (\cup_{n \in \mathbb{N}} \mathrm{LS}\{k_1, \cdots, k_n\})$ is dense in Y (Assumption F4).

We now construct a sequence K_n based on the countable set $\{k_1, k_2, \cdots\}$ described previously. Take a sequence $\varepsilon_n \to 0$ and consider the following construction:

(i) Let $K_0 = \varnothing$ and $m_0 = 0$ be the initial conditions.

(ii) For each $n \in \mathbb{N}$, let $k^{n-1}_{m_{(n-1)}+1} = k_n$ and consider the set $K_{n-1} \cup \{k^{n-1}_{m_{(n-1)}+1}\}$. Then, there are disjoint open sets $(B^n_i)^{m_n}_{i=1}$ (Fact 4) such that

   (a) $\left(\int_\Omega idv_E\right)(B^n) = \left(\int_\Omega idv_E\right)(K)$, where

$$B^n = \bigcup_{i=1}^{m_n} B^n_i,$$

   (b) $\mathrm{diam}(B^n_i) < \varepsilon_n$ for all $i = 1, \cdots, m_n$, and

(c) $k_i^{n-1} \in B_i^n$ for all $i = 1, \cdots, m_{(n-1)} + 1$.

Take arbitrary points $\left\{k^n_{m_{(n-1)}+2}, \cdots, k^n_{m_n}\right\}$ such that $k_i^n \in B_i^n$ for all $i \in \left\{m_{(n-1)} + 2, \cdots, m_n\right\}$. Finally, let $k_i^n = k_i^{n-1}$ for all $i \in \left\{1, \cdots, m_{(n-1)} + 1\right\}$ and construct the set

$$K_n = \left\{k_1^n, k_2^n, \cdots, k_{m_n}^n\right\}.$$

Notice we have $\{k_1, \cdots, k_n\} \subset K_n \in \zeta(K)$, the space of compact subsets of K which include the homogeneous commodity. Moreover, $K_n \subset K_{n+1}$ and $K_n \to K$ in closed convergence.

Consumers and Producers in $\mathcal{E}^n$

For each $n \in \mathbb{N}$, we now introduce the **approximating economy** $\mathcal{E}^n$, which is constructed by restricting $\mathcal{E}$ to the commodities in

$$K_n = \left\{k_1^n, \cdots, k_{m_n}^n\right\}.$$

For simplicity, we denote Ω_{K_n}, Y_{k_n}, X_{K_n}, P_{K_n}, and E_{K_n} by Ω_n, Y_n, X_n, P_n, and E_n, respectively.

Consumers. We need to describe the representation $e_n : I \to P_n \times E_n$ for economy $\mathcal{E}^n$, where we write $e_n(t) = (\succsim_n^t, \omega_n(t))$. Preferences will be described first, and then endowments.

1. The description of preferences $\succsim_n^t$ remains the same, namely

$$\succsim_n^t \; = \; \succsim_t \, \cap \, X_n \times X_n. \tag{14}$$

2. For the endowments, we construct a specific $\omega_n(t)$ with the following properties:

   (a) For almost every $t \in I$ and $n \in \mathbb{N}$, $\omega_n(t)(k_1) > 0$,

   (b) For every $k \in K_n$ and $n \in \mathbb{N}$, $\int \omega_n(k)\, d\lambda > 0$ and

   (c) $\omega_n(t) \to \omega(t)$ for almost every $t \in I$.

   These properties will allow us to choose the "appropriate" sequence of equilibria so that in the limit the prices are continuous.

We now describe the construction of these particular endowments. Take measurable subsets $A^n \subset I$, with $A^{n+1} \subset A^n$ for all $n \in \mathbb{N}$, and a measurable partition $\left\{A_1^n, \cdots, A_{m_n}^n\right\}$ of A^n such that

(i) $\lambda(A^n) \leq \varepsilon_n$ and

(ii) $\lambda(A_i^n) > 0$ for all $i \in \{1, \cdots, m_n\}$.

Let the functions $\eta_n : I \to E_n$ and $\xi_n : I \to E_n$ be given by

$$\eta_n(t)(k_i) = \begin{cases} \omega(t)(B_i^n) & \text{if } k_i \in K_n \\ 0 & \text{Otherwise} \end{cases}$$

and

$$\xi_n(t)(k_i) = \begin{cases} 1 & \text{if } t \in A_i^n \text{ and } k_i \in K_n \\ 0 & \text{Otherwise} \end{cases}$$

The construction of $\eta_n : I \to E_n$ is standard and follows Mas-Colell (1975) and Jones (1984). Finally, we define endowments in economy $\mathcal{E}^n$ by

$$\omega_n(t) = \eta_n(t) + \xi_n(t). \tag{15}$$

The mapping $\omega_n : I \to E_n$ is well-defined. In particular, $\omega_n(t) \in E_n$ because (a) condition (iii) in the description of economy $\mathcal{E}$ implies $\eta_n(t)\left(K^D\right) < \alpha^*$ and (b) $\xi_n(t)\left(K^D\right) \leq 1$ by construction.

Lemma 8 *$\omega_n(t) \to \omega(t)$ for almost every $t \in I$.*

Proof. Clearly, $\xi_n(t) \to 0$ for almost every $t \in I$. Hence, what remains to be proved is that $\eta_n(t) \to \omega(t)$ for almost every $t \in I$.

Denote by $A(t)$ the support of $\omega(t)$, namely

$$\begin{aligned} A(t) &= \operatorname{supp}\omega(t) \\ &= \left\{k_1^t, \cdots, k_{n_t}^t\right\}. \end{aligned}$$

For every $n \in \mathbb{N}$, we have $A(t) \subset B^n$ for almost every $t \in I$ because

$$\left(\int_\Omega i dv_E\right)(B^n) = \left(\int_\Omega i dv_E\right)(K).$$

Moreover, because λ is countably additive, it follows that for almost every $t \in I$, $A(t) \subset B^n$ for every $n \in \mathbb{N}$.

Take any open $V \subset K$ and observe

$$\omega(t)(V) = \omega(t)(V \cap A(t)).$$

For each $n \in \mathbb{N}$ and $k \in (V \cap A(t))$, there exists $B_i^n(k)$ such that $k \in B_i^n(k)$. Moreover, because $\operatorname{diam}(B_i^n) \to 0$ and V is open, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $B_i^n(k) \subset V$. Hence, we have

$$\begin{aligned} \eta_n(t)(V) &\geq \eta_n(t)\left(\bigcup_{k \in V \cap A(t)} B_i^n(k)\right) \\ &\geq \omega(t)(V). \end{aligned}$$

Therefore, we have

$$\liminf_n \eta_n(t)(V) \geq \omega(t)(V).$$

Hence, $\eta_n(t) \to \omega(t)$ for almost every $t \in I$ (Aliprantis and Border (1999), p. 276). Finally, because $\omega_n(t) = \eta_n(t) + \xi_n(t)$ and $\xi_n(t) \to 0$, we have $\omega_n(t) \to \omega(t)$ for almost every $t \in I$. ■

The endowments $\omega_n(t)$ are measurable and have the desired properties. In particular, property (a) holds because we assume all consumers are endowed with a positive quantity of the homogeneous commodity, property (b) is implied by the construction of $\xi_n : I \to E_n$, and property (c) holds by Lemma 9.

Finally, denote by v^n the probability measure on $(P_n \times E_n, \beta(P_n \times E_n))$ given by

$$v^n = \lambda \circ e_n^{-1}. \tag{16}$$

Producers. The production set Y_n and the associated aggregate production set $\mathbb{Y}_n$ for the restricted economy $\mathcal{E}^n$ are given by

$$Y_n = Y \cap \Omega_n \tag{17}$$

and

$$\mathbb{Y}_n = \left\{ y \in M(K) : y = \int_{Y_n} i \, d\sigma \ , \ \ \sigma \in M_+^1(Y_n) \right\}. \tag{18}$$

Take a positive integer $\theta^* \in \mathbb{N}$ such that $\Lambda(\mathbb{Y}_n, E_n) \subset \mathbb{Y}_n^{\theta^*}$ for all $n \in \mathbb{N}$ (because $K_n \in \zeta(K)$ for all $n \in \mathbb{N}$, Assumption BF(b) implies such integer exists). For future reference, we introduce the production sets

$$Y^{\theta^*} = \{ y \in Y : |y|(K) \leq \theta^* \}, \tag{19}$$

$$Y_n^{\theta^*} = Y_n \cap Y^{\theta^*}$$

and

$$\mathbb{Y}_n^{\theta^*} = \left\{ y \in \mathbb{Y}_n : y = \int_{Y_n^{\theta^*}} i \, d\sigma \ , \ \ \sigma \in M_+^1\left(Y_n^{\theta^*}\right) \right\}.$$

Finally, recall

$$\Lambda(\mathbb{Y}_n, E_n) = \mathbb{Y}_n \cap \left\{ y \in M(K) : y + \int_{E_n} i \, d\mu \geq 0 \ , \ \ \mu \in M_+^1(E_n) \right\}$$

and

$$\Lambda(\mathbb{Y}_n, E_n) \subset \mathbb{Y}_n^{\theta^*} \subset \mathbb{Y}^{\theta^*}.$$

Approximating Economy. Denote by $\mathcal{E}^n = [v^n, (F, Y_n)]$ the approximating economy based on $K_n = \{k_1, \cdots, k_{m_n}\}$.

3.2.2 Existence of equilibrium in economy $\mathcal{E}^n$

Denote by $M(K_n) = \{m \in M(K) : m\,|\,(K \backslash K_n) = 0\}$ the set of bounded signed Borel measures such that the restriction to the commodities in $K \backslash K_n$ is identically zero. The subspace $M(K_n)$ is isomorphic to $\mathbb{R}^{m_n}$. Therefore, with some abuse of notation, we will interpret the economy $\mathcal{E}^n$ as an economy in $\mathbb{R}^{m_n}$.

Denote by $\varphi_n : F \to M(K)$ the correspondence given by $\varphi_n(\iota) = Y_n$ for all $\iota \in F$. Denote by $S[\varphi_n]$ the set of all λ-integrable almost everywhere selections of this correspondence. For any probability measure $\sigma \in M_+^1(Y_n)$, there exists a measurable mapping $f : F \to Y_n$ such that $\sigma = \lambda \circ f^{-1}$ and the identity mapping i on Y_n is σ-integrable if and only if f is λ-integrable and $\int_{Y_n} i\, d\sigma = \int_F f\, d\lambda$. Hence, the aggregate production set of the economy can be written equivalently as $\mathbb{Y}_n = \int \varphi_n\, d\lambda$ where

$$\int \varphi_n\, d\lambda = \left\{ y \in M(K_n) : y = \int_F f\, d\lambda\ ,\ \ f \in S[\varphi_n] \right\}. \tag{20}$$

Lemma 9 *The production set $\mathbb{Y}_n$ has the properties:*

(i) $\mathbb{Y}_n$ *is closed and convex,*

(ii) $y \in \mathbb{Y}_n$ *implies* $\alpha y \in \mathbb{Y}_n$ *for all* $\alpha \in \mathbb{R}_+$ (CRS)*, and*

(iii) $0 \in \mathbb{Y}_n$ (inaction)*,* $\mathbb{Y}_n \cap \mathbb{R}_+^{m_n} = \{0\}$ (no free production)*,* $\mathbb{Y}_n \cap (-\mathbb{Y}_n) = \{0\}$ (irreversibility)*, and* $-\mathbb{R}_+^{m_n} \subset \mathbb{Y}_n$ (free disposal)*.*

Proof. Property (iii) follows directly from the assumptions on the individual and aggregate production sets. In addition, property (ii) follows from the CRS assumption on Y and the convexity of $\mathbb{Y}_n$.

We now prove property (i). The integral of a correspondence over an atomless measure space is convex (Klein and Thompson, 1984). Hence, $\mathbb{Y}_n$ is convex. To prove $\mathbb{Y}_n$ is closed, suppose $y \in M(K_n)$ is a limit point of $\mathbb{Y}_n$. Then, there exists a sequence $y_m \to y$ with $y_m \in \mathbb{Y}_n$ for all $m \in \mathbb{N}$. Moreover, there exist (a) a bounded subset $\tilde{E}_n \subset (\Omega_n \cap \Omega_+)$ and an $N \in \mathbb{N}$ such that for all $m \geq N$, we have $y_m \in \Lambda(\mathbb{Y}_n, \tilde{E}_n)$ and (b) a $\theta \in \mathbb{N}$ such that $\Lambda(\mathbb{Y}_n, \tilde{E}_n) \subset \mathbb{Y}_n^\theta$ (assumption BF(b)). Hence, for each $m \geq N$ there exists $\sigma_m \in M_+^1\left(Y_n^\theta\right)$ such that $y_m = \int_{Y_n^\theta} i\, d\sigma_m$ and, because $M_+^1\left(Y_n^\theta\right)$ is compact, the sequence $\{\sigma_m\}$ converges to some $\sigma \in M_+^1\left(Y_n^\theta\right)$ (take a subsequence if necessary). In addition, because the identity map is continuous, we have $\int_{Y_n^\theta} i\, d\sigma_m \to \int_{Y_n^\theta} i\, d\sigma$ (Fact 3). Finally, $y = \int_{Y_n^\theta} i\, d\sigma$ and $\sigma \in M_+^1\left(Y_n^\theta\right)$ implies $y \in \mathbb{Y}_n$. Hence, $\mathbb{Y}_n$ is closed. ■

Lemma 10 *For any economy $\mathcal{E}^n$ there is an equilibrium.*

Proof. Except for the indivisibilities in the consumption set, $(e_n, \mathbb{Y}_n)$ is an economy as in Hildenbrand (1974). The indivisibilities could potentially disturb the upper hemicontinuity of the individual excess demand correspondences, but we know they are upper hemicontinuous (Lemma 5).

Therefore, we can apply the existence theorem in Hildenbrand (1974, p. 219) to conclude there exists a quasi-equilibrium (a_n, y_n, p_n) of economy $(e_n, \mathbb{Y}_n)$. The free disposal assumption and the desirability of the homogeneous commodity imply we can take $p_n \in C_1^+(K_n)$. Finally, because $\omega_n(t)(h) > 0$ for almost all $t \in I$, we have $p_n \cdot \omega_n(t) > 0$ for almost all $t \in I$. Hence, we can conclude the quasi-equilibrium is an equilibrium.

Finally, because $y_n \in \Lambda(\mathbb{Y}_n, E_n) \subset \mathbb{Y}_n^{\theta^*}$, there exists $\sigma^n \in M_+^1\left(Y_n^{\theta^*}\right)$ and $f_n : F \to Y_n^{\theta^*}$ such that $y_n = \int_{Y_n^{\theta^*}} i\, d\sigma^n = \int_F f_n\, d\lambda$. Let $\tau^n = \lambda \circ (e_n, a_n)^{-1}$ be the induced probability distribution on $P_n \times E_n \times X_n$. Then, it is clear (i) $\tau^n [H_C(K_n, p_n)] = 1$, (ii) $\sigma^n [H_F(K_n, p_n)] = 1$, and (iii) $\int_{X_n} i\, d\tau_{X_n}^n = \int_{E_n} i\, d\tau_{E_n}^n + \int_{Y_n} i\, d\sigma^n$. Hence, (τ^n, σ^n, p_n) is an equilibrium of approximating economy $\mathcal{E}^n$. ■

3.3 Convergence of the sequence of equilibria

For each $n \in \mathbb{N}$, we have an equilibrium (τ^n, σ^n, p_n) of approximating economy $\mathcal{E}^n$ and equilibrium assignments $a_n : I \to X_n$ and $f_n : F \to Y_n^{\theta^*}$ such that $\tau^n = \lambda \circ (e_n, a_n)^{-1}$ and $\sigma^n = \lambda \circ (f_n)^{-1}$.

The consumption of the differentiated commodities is bounded by assumption. Moreover, because $p_n(h) = 1$ and $E_n \subset E$, if $\left\{\sup_{k \in K_n} p_n(k)\right\}_{n=1}^{\infty} \leq \eta$ for some $\eta \in \mathbb{N}$, then there exists an $\hat{\alpha} \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, we have $a_n(t)(K) \leq \hat{\alpha}$ for almost every $t \in I$. Therefore, we have

(i) σ^n is a probability measure on Y^{θ^*} and

(ii) if $\left\{\sup_{k \in K_n} p_n(k)\right\}_{n=1}^{\infty}$ is bounded, τ^n is a probability measure on $\hat{P} \times E \times \bar{X}$, where
$$\bar{X} = \{m \in X : m(K) \leq \hat{\alpha}\} \tag{21}$$
for some $\hat{\alpha} \in \mathbb{N}$.

3.3.1 Equicontinuity

For future reference, denote by $B_n(p_n, \omega_n(t))$ the **budget set** of consumer $t \in I$, namely
$$B_n(p_n, \omega_n(t)) = \{m \in X_n : p_n \cdot m \leq p_n \cdot \omega_n(t)\}.$$

Lemma 11 *$\left\{\sup_{k \in K_n} p_n(k)\right\}_{n=1}^{\infty}$ is bounded and (K_n, p_n) is equicontinuous.*

Proof. (i) $\left\{\sup_{k \in K_n} p_n(k)\right\}_{n=1}^{\infty}$ is bounded.

To show that $\left\{\sup_{k \in K_n} p_n(k)\right\}_{n=1}^{\infty}$ is bounded, suppose there is a sequence $s_n \in K_n$ such that $s_n \to s$ and $p_n(s_n) \to \infty$. Then, because $\int \omega_n(k_n)\, d\lambda > 0$ for all $k_n \in K_n$ and $n \in \mathbb{N}$, there exists a subsequence $s_{n_i} \to s$ and for each

$i \in \mathbb{N}$ (i) a subset A_{n_i} of consumers such that $\lambda(A_{n_i}) > 0$ and $a_{n_i}(t)(s_{n_i}) > 0$ for all $t \in A_{n_i}$ or (ii) a subset B_{n_i} of producers such that $\lambda(B_{n_i}) > 0$ and $f_n(\iota)(s_{n_i}) < 0$ for all $\iota \in B_{n_i}$.

Suppose (i) is true. For notational convenience, we drop the subsequence notation. Assumption BP implies there exists an $N \in \mathbb{N}$ such that

$$a_N(t) - \delta_{s_N} + p_N(s_N)\, b \succ_N^t a_N(t)$$

for all $t \in A_N$. Moreover, because $a_N(t)(s_N) > 0$ and $a_N(t) \in B_N(p_N, \omega_N(t))$, we have $a_N(t) - \delta_{s_N} + p_N(s_N)\, b \in B_N(p_N, \omega_N(t))$ for all $t \in A_N$. This is a contradiction because $a_N(t) \succsim_N^t m$ for all $m \in B_N(p_N, \omega_N(t))$ for almost every $t \in A_N$.

Alternatively, suppose (ii) is true. Then we can apply a similar argument to show this leads to a contradiction. For each $\iota \in B_n$ consider the production vector

$$\hat{f}_n(\iota) = f_n(\iota) + \delta_{s_n} - p_n(s_n)\,\gamma b$$

for some $\gamma \in (0,1)$. Then, we have $p_n \cdot \hat{f}_n(\iota) > p_n \cdot f_n(\iota)$ for all $n \in \mathbb{N}$. Moreover, assumption F3(b) implies there exists an $N \in \mathbb{N}$ such that

$$\hat{f}_N(\iota) \in Y_N$$

for all $\iota \in B_N$. This is a contradiction because $p_N \cdot f_N(\iota) \geq p_n \cdot y$ for all $y \in Y_N$ for almost every $\iota \in B_N$.

(ii) (K_n, p_n) is equicontinuous.

To show that (K_n, p_n) is equicontinuous, suppose there are sequences $s_n, t_n \in K_n$ and a real number $r > 0$ such that $s_n \to s$, $t_n \to s$ and

$$p_n(s_n) > p_n(t_n) + r$$

for all $n \in \mathbb{N}$ (take a subsequence if necessary). Then, because $\int \omega_n(k_n)\, d\lambda > 0$ for all $k_n \in K_n$ and $n \in \mathbb{N}$, there exist subsequences $s_{n_i}, t_{n_i} \to s$ and for each $i \in \mathbb{N}$ (i) a subset A_{n_i} of consumers such that $\lambda(A_{n_i}) > 0$ and $a_{n_i}(t)(s_{n_i}) > 0$ for all $t \in A_{n_i}$ or (ii) a subset B_{n_i} of producers such that $\lambda(B_{n_i}) > 0$ and $f_{n_i}(\iota)(s_{n_i}) < 0$ for all $\iota \in B_{n_k}$.

Suppose (i) is true. For notational convenience, we drop the subsequence notation. For every $n \in \mathbb{N}$, we must have $a_n(t) \in \bar{X}$ for almost every $t \in I$ (because $\left\{\sup_{k \in K_n} p_n(k)\right\}_{n=1}^{\infty}$ is bounded). Then, because $\hat{P}$ and $\bar{X}$ are compact (Lemma 3) and preferences are continuous, there exists an $N \in \mathbb{N}$ such that, for all $\succsim \in \hat{P}$ and $m \in \bar{X}$ with $m(s_n) > 0$, we have

$$m - \delta_{s_N} + \delta_{t_N} + rb \succ m.$$

Consider the consumption vector

$$\hat{a}_N(t) = a_N(t) - \delta_{s_N} + \delta_{t_N} + rb.$$

Then, we have $\hat{a}_N(t) \in B_N(p_N, \omega_N(t))$ and $\hat{a}_N(t) \succ_N^t a_N(t)$ for all $t \in A_N$. This is a contradiction because $a_N(t) \succ_N^t m$ for all $m \in B_N(p_N, \omega_N(t))$ for almost every $t \in A_N$.

Alternatively, suppose (ii) is true. Then, for each $\iota \in F_n$ consider the production vector

$$\hat{f}_n(\iota) = f_n(\iota) + \delta_{s_n} - \delta_{t_n} - rb.$$

Then, assumption F3(a) implies there exists $N \in \mathbb{N}$ such that $\hat{f}_N(\iota) \in Y_N$ and $p_N \cdot \hat{f}_N(\iota) > p_N \cdot f_n(\iota)$ for all $\iota \in B_N$. This is a contradiction because $p_N \cdot f_N(\iota) \geq p_N \cdot y$ for all $y \in Y_N$ for almost every $\iota \in B_N$. ■

3.3.2 The limit is an equilibrium of economy $\mathcal{E}$

We have τ^n is a probability measure on $\hat{P} \times E \times \bar{X}$ and σ_n is a probability measure on Y^{θ^*}. Moreover, because $\hat{P} \times E \times \bar{X}$ and Y^{θ^*} are compact, we may assume $\tau_n \to \tau$ and $\sigma_n \to \sigma$ (take a subsequence if necessary, Hildenbrand (1974, p. 49)). Finally, because the sequence (K_n, p_n) is equicontinuous, $(K_n, p_n) \to (K, p)$ for some $p \in C_1^+(K)$ (take a subsequence if necessary, Lemma 6 and Lemma 11).

In addition, because v^n is a probability measure on $\hat{P} \times E$ and $e_n(t) \to e(t)$ for almost every $t \in I$ (Lemma 8), we have $v^n \to v$ (Lemma 2 and Hildenbrand (1974, p. 51)). Finally, we know $\tau^n_{\hat{P}\times E} = v^n$, $v^n \to v$, and $\tau^n_{\hat{P}\times E} \to \tau_{\hat{P}\times E}$. Therefore, we must have $\tau_{\hat{P}\times E} = v$ (Hildenbrand (1974, p. 48)).

We will now show that (τ, σ, p) is an equilibrium of economy $\mathcal{E}$. In particular, we have

(i) $\tau^n(H_C(K_n, p_n)) = 1$ and $\tau^n \to \tau$ imply $\tau(\mathrm{Ls}(H_C(K_n, p_n))) = 1$ (Fact 6). Moreover, because $\mathrm{Ls}(H_C(K_n, p_n)) \subset H_C(K, p)$ (Lemma 5), we have

$$\tau(H_C(K, p)) = 1.$$

(ii) $\sigma^n(H_F(K_n, p_n)) = 1$ and $\sigma^n \to \sigma$ imply $\sigma(\mathrm{Ls}(H_F(K_n, p_n))) = 1$ (Fact 6). Moreover, because $\mathrm{Ls}(H_F(K_n, p_n)) \subset H_F(K, p)$ (Lemma 5), we have

$$\sigma(H_F(K, p)) = 1.$$

(iii) For each $n \in \mathbb{N}$, we have

$$\int_{X_n} i\, d\tau^n_{X_n} = \int_{E_n} i\, d\tau^n_{E_n} + \int_{Y_n} i\, d\sigma^n.$$

Moreover, $\tau^n_{X_n} \to \tau_X$, $\tau^n_{E_n} \to \tau_E$, and $\sigma^n \to \sigma$ imply $\int_{X_n} i\, d\tau^n_{X_n} \to \int_X i\, d\tau_X$, $\int_{E_n} i\, d\tau^n_{E_n} \to \int_E i\, d\tau_E$, and $\int_{Y_n} i\, d\sigma^n \to \int_Y i\, d\sigma$ (Fact 3). Therefore, we obtain

$$\int_X i\, d\tau_X = \int_E i\, d\tau_E + \int_Y i\, d\sigma.$$

Hence, (τ, σ, p) is an equilibrium of economy $\mathcal{E}$.

Appendix

This appendix contains several mathematical facts from Mas-Colell (1975) and Jones (1983). In this appendix, the following notation is used:

- K and L are complete metric spaces and $\beta(K)$ is the Borel σ-algebra of subsets of K.
- $C(K)$, $M(K)$, and $M^+(K)$ denote the sets of continuous functions, Borel measures, and non-negative Borel measures on K, respectively.
- $(Z, \beta(Z), \rho)$ is an abstract measure space and μ is a generic symbol for a measure.

Fact 1 (Mas-Colell (1975) Fact 1): Let $g : Z \to M(K)$, then $t \mapsto g(t)(V)$ is measurable for every $V \in \beta(K)$ iff $t \mapsto \int_K f\, dg(t)$ is measurable for every $f \in C(K)$.

Fact 2 (Mas-Colell (1975) Fact 2): Let $g \to M^+(K)$; if $t \mapsto g(t)(V)$ is measurable for every $V \in \beta(K)$ and $\int_Z g(t)(K)\, d\rho(t) < \infty$, then $V \mapsto \int_Z g(t)(V)\, d\rho(t)$ defines a measure (an element of $M^+(K)$), denoted by $\int_Z g\, d\rho$, which coincides with the measure induced (via the Riesz representation theorem) by the linear functional on $C(K)$, $f \mapsto \int_Z (\int_K f\, dg(t))\, d\rho(t)$.

Fact 3 (Mas-Colell (1975) Fact 3): Let K, L be compact and $g : K \to M^+(L)$ be weak-star continuous. If μ_n, $\mu \in M(K)$ and $\mu_n \to \mu$, then $\int_K g\, d\mu_n \to \int_K g\, d\mu$.

Fact 4 (Jones (1983) Lemma 3): Let K be compact and $\mu \in M^+(K)$ bounded. Then, for every $\epsilon > 0$, and every finite subset $\{t_1, \cdots, t_n\} \subset K$, there are disjoint open sets $(B_i)_{i=1}^m$ such that

(i) $\mu(\bigcup_{i=1}^m B_i) = \mu(K)$

(ii) $\text{diam}\,(B_i) < \epsilon$ for all $i = 1, \cdots, m$

(iii) $t_i \in B_i$ for all $i = 1, \cdots, n$

Fact 5 (Mas-Colell (1975) Fact 6): Let K be compact and $K_n \subset K$ closed; if $K_n \to K$ in closed convergence and $\mu \in M(K)$, then there exist $\mu_n \in M(K_n)$ such that $\mu_n \to \mu$.

Fact 6 (Jones (1983) Fact 3): Let K be compact and $K_n \subset K$ closed. If μ_n is a sequence of probability measures on K such that $\mu_n(K_n) = 1$ and $\mu_n \to \mu$, then $\mu(\text{Ls}(K_n)) = 1$.

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