A1. Proofs

A1.1. The Unregulated Bank’s Optimal Policy

Proofs of Theorem 1 and Proposition 1. We prove Theorem 1 and Proposition 1 jointly in the following two main steps. First, we prove the representation of equity and debt values and the insolvency threshold \( C_B \) in Proposition 1 given capital structure \( \theta \) and a unique switching threshold \( C_S \). Second, using these representations, we prove the uniqueness of the switching threshold \( C_S \), its characterization in (23), and the optimality of the bank’s switching policy. Proposition 1 and Theorem 1 then follow.

**Main Step 1.** In this step, we assume that the capital structure \( \theta \) is given and the bank’s project choices are determined by a unique switching threshold \( C_S \), i.e., \( P_t = 1 \) if \( C_t \geq C_S \) and \( P_t = 2 \) if \( C_t < C_S \). We then proceed to prove the representation of asset values and the insolvency threshold in Proposition 1.

By (15),

\[
S_t = E_t \int_t^{\tau_B} e^{-r(u-t)} \left[ 1_{C_u \geq \theta}(1 - \lambda_1)(C_u - \theta) + 1_{C_u < \theta}(1 + \lambda_2)(C_u - \theta) \right] du. \tag{A1}
\]

From the above, the equity value must satisfy the following flow equation (expressed in infinitesimal form for convenience)

\[
S_t = (1_{C_t \geq \theta}(1 - \lambda_1)(C_t - \theta) + 1_{C_t < \theta}(1 + \lambda_2)(C_t - \theta)) dt + E_t \left[ e^{-r dt} S_{t+dt} \right]. \tag{A2}
\]

Subtracting the L.H.S. from the R.H.S., dividing throughout by \( dt \) and eliminating terms of \( o(dt) \), we get

\[
E_t \left[ \frac{S_{t+dt} - S_t}{dt} - rS_t + (1_{C_t \geq \theta}(1 - \lambda_1)(C_t - \theta) + 1_{C_t < \theta}(1 + \lambda_2)(C_t - \theta)) \right] = 0.
\]

Applying Ito’s lemma to the above, it follows from (2) and (3) that the equity value \( S(C) \) must satisfy the following system of ODEs

\[
\begin{align*}
\frac{1}{2} \sigma_1^2 C^2 \frac{d^2 S}{dC^2} + \mu_1 C \frac{dS}{dC} - rS + (1_{C \geq \theta}(1 - \lambda_1)(C - \theta) + 1_{C < \theta}(1 + \lambda_2)(C - \theta)) &= 0 \text{ for } C > C_S \\
\frac{1}{2} \sigma_2^2 C^2 \frac{d^2 S}{dC^2} + \mu_2 C \frac{dS}{dC} - rS + (1_{C \geq \theta}(1 - \lambda_1)(C - \theta) + 1_{C < \theta}(1 + \lambda_2)(C - \theta)) &= 0 \text{ for } C < C_S.
\end{align*}
\tag{A3}
\]
(i) Suppose that $C_S > \theta$. The solution of the above system of ODEs is as follows.

$$S(C) = A_1 C_\gamma^+ + A_1' C_\gamma^+ + \frac{(1 - \lambda_1)C}{r - \mu_1} - \frac{(1 - \lambda_1)\theta}{r} \text{ for } C > C_S$$

$$= A_2 C_\gamma^- + A_3 C_\gamma'^{+} + \frac{(1 - \lambda_1)C}{r - \mu_2} - \frac{(1 - \lambda_1)\theta}{r} \text{ for } C_S > C > \theta$$

$$= A_4 C_\gamma^- + A_5 C_\gamma'^{+} + \frac{(1 + \lambda_2)C}{r - \mu_2} - \frac{(1 + \lambda_2)\theta}{r} \text{ for } C > C_B,$$

(A4)

where $\gamma_1^+, \gamma_1^-\gamma_1^-$ are the positive and negative root, respectively of (25). (It is easy to show that (25) has one positive and one negative root.) In the above, we have suppressed the dependence of the coefficients $A_1, A_1', A_2, A_3, A_4, A_5$ on $C_S$ and $\theta$.

Because $S(C) \sim \frac{(1-\lambda_1)C}{r-\mu_1}$ as $C \rightarrow \infty$, we must have $A_1' = 0$. Further, because the equity value is zero at insolvency, and insolvency is optimally chosen by the bank to maximize its equity value, it follows from well-known arguments (e.g. see Leland (1994)) that the equity value must satisfy the value matching and smooth pasting conditions (22), which ensure that the equity value is differentiable at the insolvency threshold $C_B$. The coefficients $A_1, \ldots, A_5$ and the insolvency threshold $C_B$ are determined by the conditions that the equity value must be differentiable at the switching threshold $C_S$, the debt level $\theta$ and the insolvency threshold $C_B$, that is, we have 6 unknowns and 6 equations.

In the case of $C_S < \theta$, we can use similar arguments to those used above to show that the equity value is given by

$$S_t = S(C_t) = \begin{cases} 
(\frac{(1-\lambda_1)C_t}{r} - \frac{(1-\lambda_1)\theta}{r}) + A_1(C_S, \theta)C_t\gamma_1^-, & \text{if } C_t \geq \theta, \\
(\frac{(1+\lambda_2)C_t}{r-\mu_1} - \frac{(1+\lambda_2)\theta}{r}) + A_2(C_S, \theta)C_t\gamma_1'^{+} + A_3(C_S, \theta)C_t\gamma_1^+, & \text{if } \theta > C_t \geq C_S, \\
(\frac{(1+\lambda_2)C_t}{r-\mu_2} - \frac{(1+\lambda_2)\theta}{r}) + A_4(C_S, \theta)C_t\gamma_1^- + A_5(C_S, \theta)C_t\gamma_1'^{+}, & \text{if } C_S > C_t \geq C_B(C_S, \theta). 
\end{cases}$$

(A5)

(ii) Using arguments similar to those used for the equity value, the debt value satisfies the following system of ODEs:

$$\frac{1}{2} \sigma_2^2 C^2 \frac{d^2 D}{dC^2} + \mu_1 C \frac{dD}{dC} - rD + \theta = 0 \text{ for } C > C_S$$

$$\frac{1}{2} \sigma_2^2 C^2 \frac{d^2 D}{dC^2} + \mu_2 C \frac{dD}{dC} - rD + \theta = 0 \text{ for } C < C_S.$$  

(A6)

The general solution to the above system is

$$D(C) = B_1 C\gamma_1^- + B_1' C\gamma_1'^{+} + \frac{\theta}{r} \text{ for } C > C_S$$

$$= B_2 C\gamma_2^- + B_3 C\gamma_2'^{+} + \frac{\theta}{r} \text{ for } C < C_S.$$
Since $D(C) \sim \frac{\theta}{r}$ as $C \to \infty$, $B_1' = 0$. By (10), we must have

$$D(C_B) = \frac{(1 - \lambda_1)C_B}{r - \mu_1}. \quad (A7)$$

The coefficients $B_1, B_2, B_3$ are determined by (A7) and the conditions that the debt value is differentiable at $C_S$, that is, we have 3 unknowns and 3 equations. Q.E.D.

**Main Step 2.** We next show that the optimal project selection policy is indeed given by a unique switching threshold $C_S$, which is characterized by the “super contact” condition (23). In the proof, we use the representation of equity and debt values that we have shown above in the main step 1.

We begin by stating the relevant dynamic programming verification theorem for our analysis.

**Proposition A1.** [Dynamic Programming Verification Theorem] Let $S_q(C)$ denote the equity value when the current earnings level is $C$ if the bank follows a switching policy where it chooses project 1 when its earnings exceed $q$ and project 2 when the earnings are below $q$. Suppose that $S_q(C)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\max_{i \in \{1, 2\}} L_i S_q + (1_{C \geq \theta}(1 - \lambda_1)(C - \theta) + 1_{C < \theta}(1 + \lambda_2)(C - \theta)) = 0;$$

$$S_q(C_B) = S'_q(C_B) = 0,$$  \hspace{1cm} (A8)

where

$$L_i S_q = \frac{1}{2} \sigma_i^2 C^2 \frac{d^2 S_q}{dC^2} + \mu_i C \frac{dS_q}{dC} - r S_q; i \in \{1, 2\} \quad (A9)$$

Then $S_q(C)$ is the optimal equity value function among all possible dynamic project choice policies (including non-stationary policies) and $q$ is the optimal switching trigger.

Since the above follows from the general verification theorem for dynamic programming, we omit its proof for brevity and refer the reader to Chapters 10 and 11 of Oksendal (2003), and Chapters III and IV of Fleming and Soner (2006). We use the following lemma frequently in the proof.

**A1.** We have

$$\gamma_i^+ > 1 \text{ for } i \in \{1, 2\} \quad (A10)$$

$$\gamma_i^- < \gamma_2 \quad (A11)$$

$$r - \mu_i \gamma_i^- > 0 \text{ for } i \in \{1, 2\} \quad (A12)$$

$$r - \mu_i \gamma_i^+ > 0 \text{ for } i \in \{1, 2\} \quad (A13)$$

**Proof.** By (5), and since $\gamma_i^+, \gamma_i^-$ are the roots of (25), we have $0 > \mu_i - r = \frac{1}{2} \sigma_i^2 (1)^2 + (\mu_i - \frac{1}{2} \sigma_i^2)(1) - r = \frac{1}{2} \sigma_i^2 (1 - \gamma_i^+)(1 - \gamma_i^-)$. Since $\gamma_i^- < 0$, we must have $1 < \gamma_i^+$.
Next, we note that, because $\mu_2 < \mu_1$ and $\sigma_1 < \sigma_2$,

$$\mu_2 - \frac{1}{2}\sigma_2^2 < \mu_1 - \frac{1}{2}\sigma_1^2$$

Because $\gamma^-_1 < 0$, it follows that

$$\frac{1}{2}\sigma_2^2(\gamma^-_1)^2 + (\mu_2 - \frac{1}{2}\sigma_2^2)(\gamma^-_1) - r \geq \frac{1}{2}\sigma_1^2(\gamma^-_1)^2 + (\mu_1 - \frac{1}{2}\sigma_1^2)(\gamma^-_1) - r = 0,$$

Because $\gamma^+_2, \gamma^-_2$ are the roots of $\frac{1}{2}\sigma_2^2(x)^2 + (\mu_2 - \frac{1}{2}\sigma_2^2)(x) - r = 0$, we have

$$\frac{1}{2}\sigma_2^2(\gamma^-_1)^2 + (\mu_2 - \frac{1}{2}\sigma_2^2)(\gamma^-_1) - r = \frac{1}{2}\sigma_2^2(\gamma^-_1 - \gamma^+_2)(\gamma^-_1 - \gamma^-_2) > 0.$$  

Since $\gamma^-_1 < 0 < \gamma^+_2$, it follows from the last inequality above that we must have $\gamma^-_1 < \gamma^-_2$.

To prove the third inequality, we proceed as follows.

$$0 = \frac{1}{2}\sigma_i^2(\gamma^-_i)^2 + (\mu_i - \frac{1}{2}\sigma_i^2)(\gamma^-_i) - r = \frac{1}{2}\sigma_i^2((\gamma^-_i)^2 - \gamma^-_i) + \mu_i(\gamma^-_i) - r$$

Because $\gamma^-_1 < 0$, $\frac{1}{2}\sigma_i^2((\gamma^-_i)^2 - \gamma^-_i) > 0$. Consequently, it follows from the above that (A12) must hold.

To prove the fourth inequality, observe that

$$0 = \frac{1}{2}\sigma_i^2(\gamma^+_i)^2 + (\mu_i - \frac{1}{2}\sigma_i^2)(\gamma^+_i) - r = \frac{1}{2}\sigma_i^2((\gamma^+_i)^2 - \gamma^+_i) + \mu_i(\gamma^+_i) - r$$

By (A10), $\frac{1}{2}\sigma_i^2((\gamma^+_i)^2 - \gamma^+_i) > 0$ so that inequality (A13) follows from the above. Q.E.D.

We now proceed with the proof of the main theorem. Because there are different cases to consider, we prove the theorem by stating and proving propositions that deal with each case. The following proposition establishes a necessary and sufficient condition for the bank to optimally choose project 1 always, that is, engage in no risk-shifting.

**Proposition A2.** *No Asset Substitution* Suppose that

$$L_2S_0 + (1 + \lambda_2)(C - \theta)|_{C=C_B(0)+} \leq 0,$$  \hspace{1cm} (A14)

where $S_0$ is the equity value function when the bank always chooses project 1 and $C_B(0)$ is the corresponding endogenous insolvency level. Then it is optimal for the bank to always choose project 1, that is, no asset substitution is optimal.

**Proof.** By Proposition A1, it suffices to show that

$$L_2S_0 + (1_{C \geq \theta}(1 - \lambda_1)(C - \theta) + 1_{C < \theta}(1 + \lambda_2)(C - \theta)) \leq 0$$  \hspace{1cm} (A15)

for all $C \geq C_B(0)$. 

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By Proposition 1,

\[ S_0(C) = A_1 C^{\gamma^{-}} + \frac{(1 - \lambda_1)C}{r - \mu_1} - \frac{(1 - \lambda_1)\theta}{r} \quad \text{for } C \geq \theta \]

\[ = A_2 C^{\gamma^{-}} + A_3 C^{\gamma^{+}} + \frac{(1 + \lambda_2)C}{r - \mu_1} - \frac{(1 + \lambda_2)\theta}{r} \quad \text{for } C_B(0) < C < \theta. \]  

(A16)

First, we note that

\[ A_1 > 0 \]  

(A17)

because the equity value function must be greater than \( \frac{(1 - \lambda_1)C}{r - \mu_1} - \frac{(1 - \lambda_1)\theta}{r} \), which is the equity value function in the hypothetical scenario where shareholders are not protected by limited liability.

Matching the value and derivative of the function \( S_0 \) at \( C = \theta \), we have

\[
\gamma^{-} A_1 \theta^{\gamma^{-}} + \frac{(1 - \lambda_1)\theta}{r - \mu_1} = \gamma^{+} A_2 \theta^{\gamma^{+}} + \gamma^{+} A_3 \theta^{\gamma^{+}} + \frac{(1 + \lambda_2)\theta}{r - \mu_1}.
\]  

(A18)

From the above, we obtain

\[
(\gamma^{+} - \gamma^{-}) A_3 \theta^{\gamma^{+}} = (\lambda_1 + \lambda_2) \theta \left[ \frac{\gamma^{-} - 1}{r - \mu_1} - \frac{\gamma^{-}}{r} \right] = (\lambda_1 + \lambda_2) \theta \frac{\mu_1 \gamma^{-} - r}{r(r - \mu_1)}.
\]

By (A12), and the fact that \( \gamma^{+} > \gamma^{-} \), the above implies that

\[ A_3 < 0. \]  

(A19)

Again, from (A18), we obtain

\[
(\gamma^{+} - \gamma^{-}) A_1 \theta^{\gamma^{-}} + (\lambda_1 + \lambda_2) \theta \left[ \frac{r - \mu_1 \gamma^{+}}{r(r - \mu_1)} \right] = (\gamma^{+} - \gamma^{-}) A_2 \theta^{\gamma^{-}}.
\]

By (A13) and (A17), the above implies that

\[ A_2 > 0. \]  

(A20)
Next, we note that, for $C < \theta$,

$$L_2S_0 + 1_{C<\theta}(1 + \lambda_2)(C - \theta) = A_2 \left[ \frac{1}{2} \sigma_2^2(\gamma_1^-)^2 + (\mu_2 - \frac{1}{2} \sigma_2^2)\gamma_1^- - r \right] C^{\gamma_1^-}$$

$$+ A_3 \left[ \frac{1}{2} \sigma_2^2(\gamma_1^+)^2 + (\mu_2 - \frac{1}{2} \sigma_2^2)\gamma_1^+ - r \right] C^{\gamma_1^+} + \frac{(\mu_2 - \mu_1)C}{r - \mu_1}$$

$$= A_2 \frac{1}{2} \sigma_2^2(\gamma_1^- - \gamma_2^-)(\gamma_1^- - \gamma_2^+)C^{\gamma_1^-} + A_3 \frac{1}{2} \sigma_2^2(\gamma_1^+ - \gamma_2^-)(\gamma_1^+ - \gamma_2^+)C^{\gamma_1^+}$$

$$+ (1 + \lambda_2)(\mu_2 - \mu_1)C, \quad (A21)$$

where the last equality follows from the fact that $\gamma_2^+$ and $\gamma_2^-$ are the roots of (25) for $i = 2$. Since $\gamma_1^+ < \gamma_2^- < \gamma_2^+$ by (A11), it follows from (A20) that

$$A_2 \frac{1}{2} \sigma_2^2(\gamma_1^- - \gamma_2^-)(\gamma_1^- - \gamma_2^+)C^{\gamma_1^-} > 0. \quad (A22)$$

We need to consider two cases.

**Case 1:** $\gamma_1^+ > \gamma_2^+$

By (A19),

$$A_3 \frac{1}{2} \sigma_2^2(\gamma_1^+ - \gamma_2^-)(\gamma_1^+ - \gamma_2^+)C^{\gamma_1^+} \leq 0. \quad (A23)$$

Since $\mu_2 < \mu_1$, $(1 + \lambda_2)(\mu_2 - \mu_1)C < 0$ and decreases with $C$. It then follows from (A21), (A22), and (A23) that $L_2S_0 + 1_{C<\theta}(1 + \lambda_2)(C - \theta)$ decreases with $C$. Since $L_2S_0 + (1 + \lambda_2)(C - \theta)|_{C=C_B(\theta)} < 0$ by (A14), we see that

$$L_2S_0 + (1 + \lambda_2)(C - \theta) \leq 0 \text{ for all } \theta > C \geq C_B(\theta).$$

**Case 2:** $\gamma_1^+ \leq \gamma_2^+$

In this case,

$$A_3 \frac{1}{2} \sigma_2^2(\gamma_1^+ - \gamma_2^-)(\gamma_1^+ - \gamma_2^+)C^{\gamma_1^+} \geq 0. \quad (A24)$$

By (A21), (A22), (A24), and (A10), the function $L_2S_0 + 1_{C<\theta}(1 + \lambda_2)(C - \theta)$ tends to $\infty$ as $C \to 0$ and as $C \to \infty$ and has one unique global minimum that must be negative by (A14). We will show that the function is decreasing at $C = \theta$ that, by condition (A14) would imply that it is negative in the region $\theta > C \geq C_B(\theta)$.

Observe that, for $C > \theta$

$$\frac{d}{dC}[L_1S_0 + (1 - \lambda_1)(C - \theta)] = \frac{1}{2} \sigma_1^2C^2S_0'' + \sigma_1^2CS_0'' + \mu_1S_0' + \mu_1CS_0'' - rS_0' + (1 - \lambda_1) = 0.$$
\[
\frac{d}{dC}[L_1S_0 + (1 + \lambda_2)(C - \theta)] = \frac{1}{2}\sigma_1^2 C^2 S''_0 + \sigma_1^2 C S' + \mu_1 S' + \mu_1 C S'' - rS' + (1 + \lambda_2) = 0.
\]

Since \(S_0\) is twice differentiable at \(C = \theta\), we have
\[
\frac{1}{2}\sigma_1^2 C^2 S''_0(\theta^+) - \frac{1}{2}\sigma_1^2 C^2 S''_0(\theta^-) = \lambda_1 + \lambda_2 > 0.
\]

As \(\sigma_1 < \sigma_2\),
\[
\frac{1}{2}\sigma_2^2 C^2 S''_0(\theta^+) - \frac{1}{2}\sigma_2^2 C^2 S''_0(\theta^-) > \lambda_1 + \lambda_2.
\]

Now note that, by (A25),
\[
\frac{d}{dC}[L_2S_0 + (1 - \lambda_1)(C - \theta)]\big|_{C=\theta} - \frac{d}{dC}[L_2S_0 + (1 + \lambda_2)(C - \theta)]\big|_{C=\theta^-} = \frac{1}{2}\sigma_2^2 C^2 S''_0(\theta^+) - \frac{1}{2}\sigma_2^2 C^2 S''_0(\theta^-) - (\lambda_1 + \lambda_2) > 0. \tag{A26}
\]

By (A16), for \(C > \theta\),
\[
L_2S_0 + (1 - \lambda_1)(C - \theta) = A_1 \left( \frac{1}{2}\sigma_2^2 (\gamma_1^- - \gamma_2^-)(\gamma_1^- - \gamma_2^+) C^{\gamma_1^-} + \frac{(1 - \lambda_1)(\mu_2 - \mu_1)C}{r - \mu_1} \right). 
\]

By (A11), (A17), and because \(\mu_2 < \mu_1\), the R.H.S. above decreases with \(C\), and, in particular, is decreasing at \(C = \theta^+\). By (A26), therefore, \(\frac{d}{dC}[L_2S_0 + (1 + \lambda_2)(C - \theta)]\big|_{C=\theta^-} < 0\). Consequently,
\[
L_2S_0 + (1 + \lambda_2)(C - \theta) \leq 0 \text{ for all } \theta > C \geq C_B(\theta).
\]

It remains to show that
\[
L_2S_0 + (1 - \lambda_1)(C - \theta) < 0 \text{ for all } C > \theta.
\]

Because \(S_0\) is twice differentiable at \(C = \theta\), it follows from the above that
\[
L_2S_0 + (1 - \lambda_1)(C - \theta)\big|_{C=\theta^+} < 0 
\]

We have already shown that \(L_2S_0 + (1 - \lambda_1)(C - \theta)\) is decreasing for \(C > \theta\). It follows from the above that it must be negative for \(C > \theta\).

By Proposition A1, therefore, choosing project 1 throughout is optimal for the bank. Q.E.D.

The following proposition establishes a necessary and sufficient condition for the optimal switching trigger to exceed the debt level.

**Proposition A3.** [Optimal Switching Trigger Greater than Debt Level] Suppose that
\[
L_1S_0|_{C=\theta^-} < 0, \tag{A27}
\]
where $S_\theta$ is the equity value function corresponding to the policy where the manager chooses project 2 for $C < \theta$ and project 1 for $C \geq \theta$. There exists $q^*$ with $\theta < q^* < \infty$ such that the policy of switching projects at $q^*$ is optimal for the bank and $S_{q^*}$ is the corresponding equity value function.

Proof of Proposition A3. We split the proof into several steps.

Step 1. Let $S_\infty$ be the policy of always choosing project 2. It follows using arguments similar to those used to prove Proposition 1 that

$$S_\infty(C) = A_\infty C^{\gamma_2^-} + \frac{(1 - \lambda_1)C - \theta}{r - \mu_2} \text{ for } C > \theta.$$  \hfill (A28)

Further, because equity holders are protected by limited liability, $S_\infty(C) > \frac{(1 - \lambda_1)C - \theta}{r - \mu_2}$ so that $A_\infty > 0$. Next,

$$L_1S_\infty(C) + (1 - \lambda_1)(C - \theta) = \frac{1}{2} \sigma_1^2 A_\infty (\gamma_2^- - \gamma_1^-)(\gamma_2^- - \gamma_1^+)C^{\gamma_2^-} + \frac{(1 - \lambda_1)C(\mu_1 - \mu_2)}{r - \mu_2} \gamma_2^-$$

Because, $\mu_1 > \mu_2$ and $\gamma_2^- < 0$,

$$L_1S_\infty(C) + (1 - \lambda_1)(C - \theta) \to \infty \text{ as } C \to \infty.$$ \hfill (A29)

Step 2. The function

$$\Gamma(q) = L_1S_q + (1 - \lambda_1)(C - \theta) + 1_{C\geq\theta}(1 - \lambda_1)(C - \theta) + 1_{C<\theta}(1 + \lambda_2)(C - \theta))|_{C=q^-}$$

is a continuous function of $q$. By (A29), $\Gamma(q) \to \infty$ as $q \to \infty$. By (A27), there exists $q^* \in (\theta, \infty)$ such that

$$L_1S_{q^*} + (1 - \lambda_1)(C - \theta)|_{q^*-} = 0.$$ \hfill (A30)

Since $S_{q^*}$ is the equity value function corresponding to the policy of choosing project 2 for $C < q^*$ and project 1 for $C > q^*$,

$$L_1S_{q^*} + (1 - \lambda_1)(C - \theta)|_{q^*-} = 0,$$ \hfill (A31)
$$L_2S_{q^*} + (1 - \lambda_1)(C - \theta)|_{q^*-} = 0.$$ \hfill (A32)

Subtracting (A30) from (A31), and using the fact that $S_{q^*}$ is differentiable at $q^*$, we see that

$$\frac{d^2S_{q^*}}{dC^2}|_{q=q^*} = \frac{d^2S_{q^*}}{dC^2}|_{q=q^*}.$$ \hfill (A33)

By (A32), (A32), and the fact that $S_{q^*}$ is differentiable at $q^*$, see that

$$L_2S_{q^*} + (1 - \lambda_1)(C - \theta)|_{q^*+} = 0.$$ \hfill (A34)
Step 3. We show that $q^*$ is the optimal switching trigger. By Proposition A1, we need to show that

\[ L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) \leq 0 \text{ for } C > q^* \]
\[ L_1 S_{q^*} + (1 - \lambda_1)(C - \theta) \leq 0 \text{ for } q^* \geq C > \theta \]
\[ L_1 S_{q^*} + (1 + \lambda_2)(C - \theta) \leq 0 \text{ for } \theta \geq C > C_B(q^*). \] 

(A35)

By (21), for $C > q^*$,

\[ L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) = \frac{1}{2}\sigma_2^2 A_1 C^{\gamma_1^-} (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+ \gamma_2^+ \gamma_2^- \gamma_2^+ \gamma_2^- + \frac{(1 - \lambda_1)(\mu_2 - \mu_1)C}{r - \mu_1}. \]

Since $\mu_2 < \mu_1$, the second term on the R.H.S. above is negative. By (A11), we must have $A_1 > 0$ for (A31) to hold. In this case, however, the R.H.S. of the above is a strictly decreasing function of $C$. The first condition in (A35) then follows from (A31).

Step 4. We now show that the second condition in (A35) holds. Since $S_{q^*}$ is differentiable at $q^*$,

\[ A_1(q^*)^{\gamma_1^-} + \frac{(1 - \lambda_1)q^*}{r - \mu_1} - \frac{(1 - \lambda_1)\theta}{r} = A_2(q^*)^{\gamma_2^-} + A_3(q^*)^{\gamma_2^+} + \frac{(1 - \lambda_1)q^*}{r - \mu_2} - \frac{(1 - \lambda_1)\theta}{r}, \]
\[ \gamma_1^- A_1(q^*)^{\gamma_1^-} + \frac{(1 - \lambda_1)\theta}{r - \mu_1} = \gamma_1^- A_2(q^*)^{\gamma_2^-} + \gamma_1^+ A_3(q^*)^{\gamma_2^+} + \frac{(1 - \lambda_1)q^*}{r - \mu_2}. \] 

(A36)

After some algebra, we obtain

\[ (\gamma_2^+ - \gamma_2^-) A_1(q^*)^{\gamma_1^-} + \frac{(\gamma_2^- - 1)(1 - \lambda_1)C(\mu_1 - \mu_2)}{(r - \mu_1)(r - \mu_2)} = (\gamma_2^+ - \gamma_2^-) A_2(q^*)^{\gamma_2^-}. \]

Since $\gamma_2^+ - 1 > 0$ by (A10) and $\mu_1 > \mu_2$, it follows from our earlier results that $A_1 > 0$ that

\[ A_2 > 0. \] 

(A37)

For $\theta < C < q^*$,

\[ L_1 S_{q^*} + (1 - \lambda_1)(C - \theta) = \left\{ \begin{array}{l}
\frac{1}{2}\sigma_1^2 (\gamma_2^- - \gamma_1^-) (\gamma_2^- - \gamma_1^+) C^{\gamma_2^-} + A_3 \frac{1}{2}\sigma_2^2 (\gamma_2^+ - \gamma_1^-) (\gamma_2^+ - \gamma_1^+) C^{\gamma_2^+} \\
\frac{(1 - \lambda_1)C(\mu_1 - \mu_2)}{r - \mu_2} \end{array} \right. \]

The first term on the R.H.S. above is negative by (A11) and (A37). The third term is positive because $\mu_1 > \mu_2$. There are two cases to consider.

Case 1. Suppose that the second term on the R.H.S. above is positive.

It is then easy to see that the entire expression on the R.H.S. is increasing. It follows from
(A30) that the second condition in (A35) holds.

Case 2. Suppose the second term on the R.H.S. above is negative. In this case, it follows from (A10) that the expression tends to $-\infty$ as $C \to 0$, to $-\infty$ as $C \to \infty$, and has a unique local (and global) maximum. If we show that $L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)$ is increasing to the left of $q^*$, it will follow that it is negative for $\theta < C < q^*$. We proceed as follows.

By the arguments in Step 3,

$$\frac{d}{dC} [L_2 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=q^*+} < 0 \quad (A38)$$

Since $L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) = 0$ for $\theta < C < q^*$,

$$\frac{d}{dC} [L_2 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=q^*-} = 0. \quad (A39)$$

Subtracting (A39) from (A38), evaluating the derivatives, and using the fact that $S_{q^*}$ is twice differentiable at $q^*$ by (A33), we conclude that

$$\frac{d^3}{dC^3} (S_{q^*})|_{C=q^*+} - \frac{d^3}{dC^3} (S_{q^*})|_{C=q^*-} < 0. \quad (A40)$$

Next, we note that, because $L_1 S_{q^*} + (1 - \lambda_1)(C - \theta) = 0$ for $C > q^*$,

$$\frac{d}{dC} [L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=q^*+} = 0. \quad (A41)$$

From the above, (A40), and the twice differentiability of $S_{q^*}$ at $q^*$, we see that

$$\frac{d}{dC} [L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)]|_{C=q^*-} > 0,$$

which is what we wanted to prove. It follows from (A30) that $L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)$ is negative and increasing for $\theta < C < q^*$. This establishes the second condition in (A35).

Step 5. We now show that the third condition in (A35) holds.

Since $S_{q^*}$ is differentiable at $\theta$,

$$A_2 \theta^{-2} + A_3 \theta^{-2} + \frac{(1 - \lambda_1)\theta}{r - \mu_2} - \frac{(1 - \lambda_1)\theta}{r} = A_4 \theta^{-2} + A_5 \theta^{-2} + \frac{(1 + \lambda_2)\theta}{r - \mu_2} - \frac{(1 + \lambda_2)\theta}{r},$$

$$\gamma_2^{-} A_2 \theta^{-2} + \gamma_2^{+} A_3 \theta^{-2} + \frac{(1 - \lambda_1)\theta}{r - \mu_2} = \gamma_2^{-} A_4 \theta^{-2} + \gamma_2^{+} A_5 \theta^{-2} + \frac{(1 + \lambda_2)\theta}{r - \mu_2}. \quad (A42)$$

After some algebra, we can show that

$$\left(\gamma_2^{+} - \gamma_2^{-}\right) A_2 \theta^{-2} + (\lambda_1 + \lambda_2) \left[\frac{r - \gamma_2^{+} \mu_2}{r(r - \mu_2)}\right] = \left(\gamma_2^{+} - \gamma_2^{-}\right) A_4 \theta^{-2}.$$ 

The first term on the L.H.S. above is positive by (A37). The second term is also positive by
Consequently,
\[ A_4 > 0. \]  \hspace{1cm} (A43)

For \( C < \theta \),
\[
L_1S_q^* + (1 + \lambda_2)(C - \theta) = A_4 \frac{\sigma_1^2(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_1^+)}{r - \mu_2} + A_5 \frac{\sigma_2^2(\gamma_2^+ - \gamma_1)(\gamma_2^+ - \gamma_1^+)}{r - \mu_2} \]

The first term on the R.H.S. above is negative by (A11) and (A43). The third term is positive because \( \mu_1 > \mu_2 \). There are again two cases to consider.

**Case 1.** Suppose that the second term on the R.H.S. above is positive.

It is then easy to see that the entire expression on the R.H.S. is increasing. By the arguments in Step 4,
\[
L_1S_q^* + (1 + \lambda_2)(C - \theta) |_{C=\theta^+} < 0.
\]

Since \( S_q^* \) is twice differentiable at \( C = \theta \),
\[
L_1S_q^* + (1 + \lambda_2)(C - \theta) |_{C=\theta^-} < 0. \hspace{1cm} (A44)
\]

Since \( L_1S_q^* + (1 + \lambda_2)(C - \theta) \) is increasing for \( C < \theta \), it is negative for \( C < \theta \).

**Case 2.** Suppose the second term on the R.H.S. above is negative. In this case, it follows from (A10) that the expression tends to \(-\infty\) as \( C \to 0 \), to \(-\infty\) as \( C \to \infty \), and has a unique local (and global) maximum. If we show that \( L_1S_q^* + (1 - \lambda_1)(C - \theta) \) is increasing to the left of \( \theta \), it will follow that it is negative for \( C < \theta \). We proceed as follows.

First, note that, because
\[
L_2S_q^* + (1 - \lambda_1)(C - \theta) = 0 \text{ for } C > \theta, \quad L_2S_q^* + (1 + \lambda_2)(C - \theta) = 0 \text{ for } C < \theta,
\]

\[
\frac{d}{dC} L_2S_q^* |_{C=\theta^+} - \frac{d}{dC} L_2S_q^* |_{C=\theta^-} = \lambda_1 + \lambda_2 > 0.
\]

Since \( S_q^* \) is twice differentiable at \( \theta \),
\[
\frac{d^2}{dC^2} L_2S_q^* |_{C=\theta^+} - \frac{d^2}{dC^2} L_2S_q^* |_{C=\theta^-} = \frac{1}{2} \sigma_2^2 \left[ \frac{d^3}{dC^3} (S_q^*) |_{C=\theta^+} - \frac{d^3}{dC^3} (S_q^*) |_{C=\theta^-} \right] > 0.
\]
Since $\sigma_2 > \sigma_1$

$$\frac{1}{2} \sigma_2^2 \left[ \frac{d^3}{dC^3} (S_{q^*}) |_{C=\theta^+} - \frac{d^3}{dC^3} (S_{q^*}) |_{C=\theta^-} \right] > \frac{1}{2} \sigma_1^2 \left[ \frac{d^3}{dC^3} (S_{q^*}) |_{C=\theta^+} - \frac{d^3}{dC^3} (S_{q^*}) |_{C=\theta^-} \right]$$

$$= \frac{d}{dC} L_1 S_{q^*} |_{C=\theta^+} - \frac{d}{dC} L_1 S_{q^*} |_{C=\theta^-}$$

It follows from the above that

$$\frac{d}{dC} L_1 S_{q^*} |_{C=\theta^+} - \frac{d}{dC} L_1 S_{q^*} |_{C=\theta^-} - (\lambda_1 + \lambda_2) < 0$$

By the results of Step 4, $\frac{d}{dC} [L_1 S_{q^*} + (1 - \lambda_1)(C - \theta^+)] |_{C=\theta^+} > 0$. It follows from the above that

$$\frac{d}{dC} [L_1 S_{q^*} + (1 + \lambda_2)(C - \theta^-)] |_{C=\theta^-} > 0,$$

which is exactly what we wanted to prove. It follows from (A44) that $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta) < 0$ for $C < \theta$.

In conclusion, we have shown that all three conditions of (A35) hold and, moreover, the inequalities are strict. Consequently, by Proposition A1, the policy of switching projects at $q^*$ is the unique optimal policy for the bank. Q.E.D.

The following proposition completes the remaining step in the proof of the theorem by establishing conditions under which the optimal switching trigger is less than the debt level.

**Proposition A4.** [Optimal Switching Trigger Less than Debt Level] Suppose that

$$L_1 S_{q^*} |_{C=\theta^-} \geq 0, L_2 S_0 + (1 + \lambda_2)(C - \theta) |_{C=\theta^-} > 0. \quad (A45)$$

There exists $q^* \leq \theta$ such that the policy of switching projects at $q^*$ is optimal for the bank and $S_{q^*}$ is the corresponding equity value function.

**Proof of Proposition A4.** We again split the proof up into several steps.

**Step 1.** Suppose first that $L_1 S_0 |_{C=\theta^-} > 0$. By (A45), It follows as a special case of Proposition A1 that, within the sub-class of policies characterized by a single switching trigger that is less than $\theta$, the policy of always choosing project 1 as well as the policy of switching at $C = \theta$ are both sub-optimal. By the continuity of the bank’s objective function in the switching trigger, it follows that there exists $q^* < \theta$ such that the policy of switching projects at $q^*$ is optimal within the restricted sub-class of policies characterized by a single switching trigger. We will show that the policy of switching projects at $q^*$ is, in fact, **globally optimal** among all possible dynamic project choice policies.

**Step 2.** We now show that

$$L_1 S_{q^*} + (1 + \lambda_2)(C - \theta) |_{q^*-} = L_2 S_{q^*} + (1 + \lambda_2)(C - \theta) |_{q^*+} = 0. \quad (A46)$$
Suppose to the contrary that $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*-} < 0$. We can use Ito’s lemma to show that there exists $q^{**} > q^*$ such that the value of the policy of switching projects at $q^{**}$ is greater than the corresponding value for switching projects at $q^*$. Suppose, on the other hand, that $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*+} > 0$. In this case, we can show that there exists $q^{**} < q^*$ such that the value of the policy of switching projects at $q^{**}$ is greater than the corresponding value for switching projects at $q^*$. In either case, the policy of switching projects at $q^*$ is suboptimal within the sub-class of policies characterized by a single switching trigger, which is a contradiction. Hence, $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*} = 0$. Since the bank chooses project 1 for $C > q^*$, $L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*+} = 0$. It follows that $L_2 S_{q^*} + (1 + \lambda_2)(C - \theta)|_{q^*+} = 0$. Moreover, the super contact condition (A33) holds at $q^*$, that is, the value function is twice differentiable at $q^*$.

**Step 3.** By Proposition A1, we need to show that

$$
L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) \leq 0 \text{ for } C > \theta \\
L_2 S_{q^*} + (1 + \lambda_2)(C - \theta) \leq 0 \text{ for } \theta \geq C > q^* \\
L_1 S_{q^*} + (1 + \lambda_2)(C - \theta) \leq 0 \text{ for } q^* \geq C > C_B(q^*). 
$$

(A47)

By (A5),

$$
L_2 S_{q^*} + (1 - \lambda_1)(C - \theta) = \frac{1}{2} \sigma^2 A^1 C^{n^1} (\gamma^1 - \gamma^2_1) (\gamma^1 - \gamma^2_2) + \frac{(1 - \lambda_1)(\mu_2 - \mu_1)C}{r - \mu_1} 
$$

(A48)

By the arguments in Step 1, $S_{q^*}$ is strictly greater than the equity value from the policy of always choosing project 1. The latter, in turn, exceeds $\frac{(1 - \lambda_1)C}{r - \mu_1}$ because equity is protected by limited liability. Consequently,

$$
A_1 > 0. 
$$

(A49)

It follows from (A11) and the fact that $\mu_1 > \mu_2$ that both terms on the R.H.S. of (A48) are decreasing. Hence, $L_2 S_{q^*} + (1 - \lambda_1)(C - \theta)$ decreases for $C > \theta$. Next, observe that, for $C > \theta$

$$
\frac{d}{dC}[L_1 S_{q^*} + (1 - \lambda_1)(C - \theta)] = \frac{1}{2} \sigma^2 C^2 S_{q^*}''' + \sigma^2 C S_{q^*}'' + \mu_1 S_{q^*}' + \mu_1 C S_{q^*}'' - r S_{q^*}' + (1 - \lambda_1) = 0.
$$

For $q^* < C < \theta$

$$
\frac{d}{dC}[L_1 S_{q^*} + (1 + \lambda_2)(C - \theta)] = \frac{1}{2} \sigma^2 C^2 S_{q^*}''' + \sigma^2 C S_{q^*}'' + \mu_1 S_{q^*}' + \mu_1 C S_{q^*}'' - r S_{q^*}' + (1 + \lambda_2) = 0.
$$

Since $S_{q^*}$ is twice differentiable at $C = \theta$, we have

$$
\frac{1}{2} \sigma^2 C^2 S_{q^*}'''(\theta+) - \frac{1}{2} \sigma^2 C^2 S_{q^*}'''(\theta-) = \lambda_1 + \lambda_2 > 0.
$$
As $\sigma_1 < \sigma_2$,
\[
\frac{1}{2}\sigma_2^2C^2S_q''(\theta^+) - \frac{1}{2}\sigma_2^2C^2S_q''(\theta^-) > \lambda_1 + \lambda_2. \tag{A50}
\]

Now note that, by (A25),
\[
\frac{d}{dC}[L_2S_q^* + (1 - \lambda_1)(C - \theta)]|_{C=\theta^+} - \frac{d}{dC}[L_2S_q^* + (1 + \lambda_2)(C - \theta)]|_{C=\theta^-} = \frac{1}{2}\sigma_2^2C^2S_q''(\theta^+) - \frac{1}{2}\sigma_2^2C^2S_q''(\theta^-) > 0. \tag{A51}
\]

We have already shown earlier that $\frac{d}{dC}[L_2S_q^* + (1 - \lambda_1)(C - \theta)]|_{C=\theta^+} < 0$. Consequently, $\frac{d}{dC}[L_2S_q^* + (1 + \lambda_2)(C - \theta)]|_{C=\theta^-} < 0$. Therefore, we have shown that $L_2S_q^* + 1_{C<\theta}(1 + \lambda_2)(C - \theta) + 1_{C>\theta}(1 + \lambda_2)(C - \theta)$ is strictly decreasing for all $C > q^*$. It then follows from (A46) that it must be negative for all $C > q^*$. Hence, we have established the first two conditions in (A47).

It remains to establish the third condition.

**Step 4.** By our previous arguments,
\[
\frac{d}{dC}[L_2S_q^* + (1 + \lambda_2)(C - \theta)]|_{C=q^*^+} < 0. \tag{A52}
\]

Since the bank chooses project 2 for $C < q^*$,
\[
\frac{d}{dC}[L_2S_q^* + (1 + \lambda_2)(C - \theta)]|_{C=q^*-} = 0. \tag{A53}
\]

Subtracting (A53) from (A52), we get
\[
S_q''(q^*)^+ - S_q''(q^*)^- > 0. \tag{A54}
\]

Next, because the bank chooses project 1 for $C > q^*$,
\[
\frac{d}{dC}[L_1S_q^* + (1 + \lambda_2)(C - \theta)]|_{C=q^*-} = 0. \tag{A55}
\]

By (A54) and (A55),
\[
\frac{d}{dC}[L_1S_q^* + (1 + \lambda_2)(C - \theta)]|_{C=q^*-} > 0.
\]

We can use arguments similar to those we have used previously in the proof to show that the above condition implies that $L_1S_q^* + (1 + \lambda_2)(C - \theta)$ is strictly increasing for $C < q^*$. It then follows from (A46) that it is strictly negative for $C < q^*$, which establishes the third condition in (A47).

**Step 5.** Suppose first that $L_1S_q|_{C=\theta^-} = 0$. We can use arguments very similar to those used above to show that the policy of switching projects at $\theta$ is optimal. We omit the arguments for brevity. Q.E.D.
The second main step in the proof of Proposition 1 and Theorem 2 now follow from Propositions A2, A3 and A4. Q.E.D.

### A1.2. The Optimal Regulatory Policy

#### Proofs of Theorem 2 and Proposition 2.

Similar to the case of unregulated bank, our proof consists of two main steps. First, we prove the representation of social, equity and debt values and the bailout threshold \( C^{'reg}_B \) in Proposition 2 given capital structure \( \theta^{reg} \) and a unique switching threshold \( C^{reg}_S \). Second, using these representations, we prove the uniqueness of the switching threshold \( C^{reg}_S \) and that it satisfies \( C^{reg}_S \leq \theta^{reg} \) and the supercontact condition

\[
F^{social''}(C^{reg}_S +) = F^{social''}(C^{reg}_S -). \tag{A56}
\]

Proposition 2 and Theorem 2 then follow. For brevity of notation, in the remaining of this section we omit the superscript “\( \text{reg} \)” in the regulator’s policy.

**Main Step 1.** In this step, we assume that the capital structure \( \theta \) is given and that the regulator’s project choices are determined by a unique switching threshold \( C_S \), i.e., \( P_t = 1 \) if \( C_t \geq C_S \) and \( P_t = 2 \) if \( C_t < C_S \). The following proposition provide the detailed representation of social, equity, and debt values and the bailout threshold \( C_B \).

**Proposition A5.** [Social, Equity, and Debt Values and Bailout Threshold] Suppose the debt level is \( \theta \) and the regulator adopts a policy where it chooses project 1 if \( C_t \geq C_S \) and project 2 for \( C_t < C_S \), then there exists a bailout threshold \( C_B \) such that the following holds.

(i) In the case \( C_S \geq \theta \), the bank’s social value at any date \( t \) is a continuously differentiable function of the current earnings level \( C_t \) and is given by

\[
F^{social}_t = \begin{cases} 
\frac{(1 - \lambda)}{r - \mu_1} C_t + \frac{\lambda \theta}{r} + X_1 C_t^{\gamma_1}, & \text{if } C_t \geq C_S, \\
\frac{(1 - \lambda)}{r - \mu_2} C_t + \frac{\lambda \theta}{r} + X_2 C_t^{\gamma_2} + X_3 C_t^{\gamma_2^+}, & \text{if } C_S > C_t \geq \theta, \\
\frac{(1 + \lambda \gamma)}{r - \mu_2} C_t - \frac{\lambda \theta}{r} + X_4 C_t^{\gamma_2} + X_5 C_t^{\gamma_2^+}, & \text{if } \theta > C_t > C_B, \\
\frac{(1 - \omega)(1 - \lambda)}{r - \mu_1} C_B, & \text{if } C_t = C_B.
\end{cases} \tag{A57}
\]

Furthermore, the social value satisfies the following value matching and smooth pasting boundary conditions at the bailout threshold \( C_B \),

\[
F^{social'}(C_B) = \frac{(1 - \omega)(1 - \lambda)}{r - \mu_1}. \tag{A58}
\]

The coefficients \( X_i; i = 1, \ldots, 5 \) in the above are determined by the conditions that the social value is continuously differentiable throughout. The bailout threshold, \( C_B \), is determined by the smooth pasting condition (A58) for the social value at the bailout threshold.
(ii) In the case \( C_s < \theta \),

\[
F_t^{social} = F^{social}(C_t) = \begin{cases} 
(1-\lambda)C_t + \frac{\lambda \theta}{r} + X_1 C_t^{\gamma_1}, & \text{if } C_t \geq \theta, \\
(1+\lambda_2)C_t - \frac{\lambda_2 \theta}{r} + X_2 C_t^{\gamma_1} + X_3 C_t^{\gamma_1^+}, & \text{if } \theta > C_t \geq C_s, \\
(1+\lambda_2)C_t - \frac{\lambda_2 \theta}{r} + X_4 C_t^{\gamma_2} + X_5 C_t^{\gamma_2^+}, & \text{if } C_s > C_t > C_B, \\
(1-\omega)(1-\lambda)C_B - \frac{\mu_1}{r}, & \text{if } C_t = C_B. 
\end{cases}
\]  

(A59)

and the social value satisfies the same smooth-pasting condition in (A58).

(iii) The bank’s equity value at any date \( t \) is given by (21) where the switching trigger is \( C_s \). The endogenous insolvency threshold at which the equity value falls to zero is \( C_B \), which is determined by the smooth pasting condition (22).

(iv) The bank’s debt value at any date \( t \) is

\[
D_t = D(C_t) = \begin{cases} 
\frac{\theta}{r} + B_1 C_t^{\gamma_1}, & \text{if } C_t \geq C_s, \\
\frac{\theta}{r} + B_2 C_t^{\gamma_2} + B_3 C_t^{\gamma_2^+}, & \text{if } C_s > C_t > C_B, \\
(1-\omega)(1-\lambda)C_B - \frac{\mu_1}{r}, & \text{if } C_t = C_B. 
\end{cases}
\]  

(A60)

We omit the proof of the Proposition because it follows using exactly the same arguments used to prove Proposition 1.

**Main Step 2:** We next show that the regulator’s optimal project selection policy is indeed given by a unique switching threshold \( C_s \), which is characterized by the “super contact” condition (A56). In the proof, we shall use the representation of equity and debt values that we have shown above in the main step 1. As in the proof of Theorem 1, we proceed by stating and proving intermediate propositions. The following proposition states the relevant dynamic programming verification theorem.

**Proposition A6.** [Verification Theorem for Regulator’s Problem] Let \( F_q^{social}(C) \) denote the social value when the current earnings level is \( C \) if the regulator follows a switching policy where it switches projects at the earnings level \( q \). Suppose that \( F_q^{social}(C) \) satisfies the following HJB equation:

\[
\max_{i \in \{1,2\}} \mathcal{L}_i F_q^{social}(C) + [1_{C \geq \theta} \{(1-\lambda)C + \lambda \theta\} + 1_{C < \theta} \{(1+\lambda_2)C - \lambda_2 \theta\}] = 0;
\]

\[
F_q^{social}(C_B) = \frac{(1-\omega)(1-\lambda)C_B}{r - \mu_1};
\]

\[
F_q^{social'}(C_B) = \frac{(1-\omega)(1-\lambda)}{r - \mu_1},
\]

(A61)

(A62)

Then \( F_q^{social}(C) \) is the optimal social value function among all possible dynamic project choice.
policies (including non-stationary policies) and \( q \) is the optimal switching trigger.

The following proposition establishes the necessary and sufficient condition for the regulator to optimally choose project 1 throughout, that is, engage in no risk-shifting.

**Proposition A7. [No Asset Substitution for Regulator]** Suppose that

\[
L_2 F_0^{social} + (1 + \lambda_2)C - \lambda_2 \theta |C = C_B(0) + 0, \tag{A63}
\]

where \( F_0^{social} \) is the social value function when the regulator always chooses project 1 and \( C_B(0) \) is the corresponding endogenous bailout level. Then it is optimal for the regulator to always choose project 1, that is, no asset substitution is optimal.

**Proof.** By (A59), the social value when the regulator always chooses project 1 is given by

\[
F_0^{social}(C) = X_1 C_{\gamma_1^-} + \frac{(1 - \lambda)C}{r - \mu_1} + \frac{\lambda \theta}{r} \text{ for } C \geq \theta
\]

\[
= X_2 C_{\gamma_2^-} + X_3 C_{\gamma_1^{+} - \gamma_2^-} + \frac{(1 + \lambda_2)C}{r - \mu_1} - \frac{\lambda \theta}{r} \text{ for } C < \theta. \tag{A64}
\]

The social value function in the hypothetical scenario where the bank’s debt is completely risk-free is \( (1 - \lambda)C + \frac{\lambda \theta}{r} \). Since \( F_0^{social}(C) \) must clearly be less, we must have

\[
X_1 < 0. \tag{A65}
\]

Next, we note that, for \( C > \theta \),

\[
L_2 F_0^{social} + (1 - \lambda)C + \lambda \theta = \frac{1}{2} \sigma_2^2 X_1 (\gamma_1^- - \gamma_2^-)(\gamma_1^- - \gamma_2^-)C_{\gamma_1^-} + \frac{(1 - \lambda)C(\mu_2 - \mu_1)}{r - \mu_1}.
\]

Because \( \gamma_1^- < \gamma_2^- \) by (A11), it follows from (A65) that the first term on the R.H.S. above is strictly negative. The second term is also strictly negative because \( \mu_2 < \mu_1 \). Consequently, \( L_2 F_0^{social} + (1 - \lambda)C + \lambda \theta < 0 \), which establishes the verification condition for \( C > \theta \).

Since the social value function is differentiable at \( \theta \),

\[
X_1 \theta_{\gamma_1^-} + \frac{(1 - \lambda)\theta}{r - \mu_1} + \frac{\lambda \theta}{r} = X_2 \theta_{\gamma_2^-} + X_3 \theta_{\gamma_1^{+} - \gamma_2^-} + \frac{(1 + \lambda_2)\theta}{r - \mu_1} - \frac{\lambda \theta}{r}
\]

\[
\gamma_1^- X_1 \theta_{\gamma_1^-} + \frac{(1 - \lambda)\theta}{r - \mu_1} = \gamma_1^- X_2 \theta_{\gamma_2^-} + \gamma_1^{+} X_3 \theta_{\gamma_1^{+} - \gamma_2^-} + \frac{(1 + \lambda_2)\theta}{r - \mu_1}
\]

From the above, we obtain

\[
(\gamma_1^{+} - \gamma_1^-)X_3 \theta_{\gamma_1^{+}} = (\lambda + \lambda_2)\theta \left[ \frac{\gamma_1^- - 1}{r - \mu_1} - \frac{\gamma_1^-}{r} \right] = (\lambda + \lambda_2)\theta \frac{\mu_1 \gamma_1^- - r}{r(r - \mu_1)}.
\]
By (A12), and the fact that \( \gamma_1^+ > \gamma_1^- \), the above implies that
\[
X_3 < 0. \tag{A66}
\]

Next, note that, for \( C < \theta \),
\[
L_2 F_0^{social} + (1 + \lambda_2)C - \lambda_2 \theta = \frac{1}{2} \sigma_2^2 X_2 (\gamma_1^- - \gamma_2^-) (\gamma_1^- - \gamma_2^+) C^{\gamma_1^-} + \frac{1}{2} \sigma_2^2 X_3 (\gamma_1^+ - \gamma_2^-) (\gamma_1^+ - \gamma_2^+) C^{\gamma_1^+} + \frac{(1 - \lambda)C (\mu_2 - \mu_1)}{\mu - \mu_1}. \tag{A67}
\]

Because \( F_0^{social} \) is twice differentiable at \( C = \theta \), it follows from our earlier result that \( L_2 F_0^{social} + (1 - \lambda)C + \lambda \theta < 0 \) for \( C > 0 \) that
\[
L_2 F_0^{social} + (1 + \lambda_2)C - \lambda_2 \theta|_{C=\theta^-} = L_2 F_0^{social} + (1 - \lambda)C + \lambda \theta|_{C=\theta^+} < 0 \tag{A68}
\]

We need to consider two cases.

Case 1: \( \gamma_1^+ \geq \gamma_2^+ \).

In this case, the second term on the right hand side of (A67) is non-positive by (A66). The third term is negative because \( \mu_2 < \mu_1 \). If \( X_2 \leq 0 \), then the first term is non-positive by (A11) so that the entire expression on the R.H.S. of (A67) is negative for \( C < \theta \). If \( X_2 > 0 \), then the expression on the R.H.S. of (A67) is decreasing with \( C \). In this case, condition (A63) implies that it is again negative for \( C_B(0) < C < \theta \).

Case 2: \( \gamma_1^+ < \gamma_2^+ \)

In this case, the second term on the right hand side of (A67) is positive by (A66). If \( X_2 < 0 \), then the first term is positive by (A11). The function on the R.H.S. of (A67) tends to \( \infty \) as \( C \rightarrow 0 \) and as \( C \rightarrow \infty \) and has a unique local (and global) minimum that is strictly negative by (A63). Condition (A63) and (A68) together imply that \( L_2 F_0^{social} + (1 + \lambda_2)C - \lambda_2 \theta \) must, in fact, be negative for \( C_B(0) < C < \theta \).

In summary, we have shown that \( F_0^{social} \) satisfies the HJB equation. Moreover, \( L_2 F_0^{social}(C) + \left[ 1_{C_B(0) < C < \infty} \left\{ (1 - \lambda)C + \lambda \theta \right\} + 1_{C < C_B(0)} \left\{ (1 + \lambda_2)C - \lambda_2 \theta \right\} \right] < 0 \) for \( C_B(0) < C < \infty \), which implies that choosing project 1 always is the unique optimal policy for the regulator. Q.E.D.

The following proposition shows that, if condition (A63) does not hold, the regulator optimally switches projects at a trigger that is below the debt level.

**Proposition A8. [Optimal Switching Trigger]** Suppose that
\[
L_2 F_0^{social} + (1 + \lambda_2)C - \lambda_2 \theta|_{C=C_B(0)+} > 0. \tag{A69}
\]

There exists \( q^* < \theta \) such that it is optimal for the regulator to choose project 2 for \( C < q^* \), and project 1 for \( C \geq q^* \).

**Proof.** We split the proof into several steps.
Step 1. Consider the policy where the switching trigger is equal to \( \theta \). By (A57), the social value function for \( C > \theta \) has the form

\[
F_{social}^{\theta}(C) = X_1 C^{\gamma_1^-} + \frac{(1 - \lambda) C}{r - \mu_1} + \frac{\lambda \theta}{r}.
\]

By the same argument used in the proof of Proposition A7,

\[X_1 < 0.\]  \(\text{(A70)}\)

Consequently, for \( C > \theta \),

\[
L_2 F_{social}^{\theta} + (1 - \lambda) C + \lambda \theta = \frac{1}{2} \sigma^2 \tau X_1 (\gamma_1^- - \gamma_2^-)(\gamma_1^- - \gamma_2^+)C^{\gamma_1^-} + \frac{(1 - \lambda) C (\mu_2 - \mu_1)}{r - \mu_1} < 0, \quad (A71)
\]

so that

\[
L_2 F_{social}^{\theta} + (1 - \lambda) C + \lambda \theta|_{C=\theta^+} < 0. \quad (A72)
\]

We know that, because project 2 is chosen for \( C < \theta \), and project 1 for \( C > \theta \),

\[
L_2 F_{social}^{\theta} + (1 + \lambda_2) C - \lambda_2 \theta|_{C=\theta^-} = 0, \\
L_1 F_{social}^{\theta} + (1 - \lambda_1) C + \lambda_1 \theta|_{C=\theta^+} = 0.
\]

From the above equations and (A72), we obtain

\[
L_1 F_{social}^{\theta} + (1 + \lambda_2) C - \lambda_2 \theta|_{C=\theta^-} > 0. \quad (A73)
\]

By (A69) and (A73), the policy of always choosing project 1, and the policy of switching projects at \( \theta \) are both sub-optimal within the restricted class of “single switching trigger” policies where the switching trigger is less than or equal to \( \theta \). By the continuity of the regulator’s objective function, there exists \( q^* < \theta \) such that the policy of switching projects at \( q^* \) is optimal within the restricted class of “single switching trigger” policies. We will show that the policy is, in fact, globally optimal as in the proof of Proposition A4.

Step 2. By arguments similar to those used in Step 2 of the proof of Proposition A4, we must have

\[
L_2 F_{q^*}^{social} + (1 + \lambda_2) C - \lambda_2 \theta|_{C=q^+} = 0, \quad (A74)
\]

\[
L_1 F_{q^*}^{social} + (1 + \lambda_2) C - \lambda_2 \theta|_{C=q^-} = 0. \quad (A75)
\]

Further, the above imply that \( F_{q^*}^{social} \) is twice differentiable at \( q^* \).

Step 3. Using arguments similar to those used in the proof of Proposition A7, we can show that, condition (A74) implies that \( L_2 F_{q^*}^{social} + (1 + \lambda_2) C - \lambda_2 \theta < 0 \) for \( q^* < C < \theta \).
Step 4. It remains to show that \( L_1 F_{q^{social}} + (1 + \lambda_2)C - \lambda_2 \theta < 0 \) for \( C_B(q^*) < C < q^* \). By arguments similar to those used in Step 4 of the proof of Proposition A4, we can show that \( L_1 F_{q^{social}} + (1 + \lambda_2)C - \lambda_2 \theta \) is increasing for \( C < q^* \). Condition (A75), therefore, implies that it is negative for \( C < q^* \).

Hence, the function \( F_{q^{social}} \) satisfies all the conditions of the verification theorem. Moreover, the results that

\[
L_2 F_{q^{social}}(C) + [1_{C \geq \theta} \{(1 - \lambda)C + \lambda \theta\} + 1_{C < \theta} \{(1 + \lambda_2)C - \lambda_2 \theta\}] < 0 \quad \text{for} \quad C > q^*,
\]

\[
L_1 F_{q^{social}}(C) + [1_{C \geq \theta} \{(1 - \lambda)C + \lambda \theta\} + 1_{C < \theta} \{(1 + \lambda_2)C - \lambda_2 \theta\}] < 0 \quad \text{for} \quad C < q^*,
\]

together imply that switching projects at \( q^* \) is the unique optimal policy for the regulator. This completes main step 2 and the proofs of Theorem 2 and Proposition 2. Q.E.D.

**Proof of Proposition 3.**

We first show that the condition

\[
\gamma_1^+ < 1 + \frac{2(\mu_1 - \mu_2)}{\sigma_2^2 - \sigma_1^2}
\]  

(A76)

is equivalent to \( \gamma_1^+ < \gamma_2^+ \). Since \( \gamma_1^+ > 0 > \gamma_2^- \),

\[
\frac{1}{2} \sigma_2^2 \gamma_1^+ \gamma_2^+ + (\mu_2 - \frac{1}{2} \sigma_2^2) \gamma_1^+ - r = \frac{1}{2} \sigma_2^2 (\gamma_1^+ - \gamma_2^+) (\gamma_1^+ - \gamma_2^-) \tag{A77}
\]

has the same sign as \( \gamma_1^+ - \gamma_2^+ \). On the other hand, since \( \frac{1}{2} \sigma_1^2 \gamma_1^+ \gamma_2^+ + (\mu_1 - \frac{1}{2} \sigma_1^2) \gamma_1^+ - r = 0 \),

\[
\frac{1}{2} \sigma_2^2 \gamma_1^+ \gamma_2^+ + (\mu_2 - \frac{1}{2} \sigma_2^2) \gamma_1^+ - r = \frac{1}{2} (\sigma_2^2 - \sigma_1^2) \gamma_1^+ \gamma_2^+ + (\mu_2 - \mu_1 - \frac{1}{2} (\sigma_2^2 - \sigma_1^2)) \gamma_1^+ \\
= \frac{1}{2} (\sigma_2^2 - \sigma_1^2) [\gamma_1^+ + \frac{2(\mu_2 - \mu_1)}{\sigma_2^2 - \sigma_1^2} - 1]. \tag{A78}
\]

Given that \( \sigma_2 > \sigma_1 \), it follows from (A77) and (A78) that (A76) is equivalent to \( \gamma_1^+ < \gamma_2^+ \).

We proceed below with the proof of (27). To simplify the notation, let \( q^* \) and \( q^*_r \) denote the optimal switching triggers in the bank’s and regulator’s problems, respectively. We need to show that \( q^*_r < q^* \). Since the regulator’s optimal switching trigger is less than \( \theta \) by Theorem 2, it suffices to consider the case where \( q^* < \theta \). We prove the proposition in two main parts.

**Part 1.** Let \( S^{social} \) be the social value of equity with cost parameters \( (\lambda, \lambda_2) \). We will show that the optimal switching point \( q^{social}_1 \) is lower than \( q^* \).

Consider a switching policy described by a trigger \( q \), where \( q \geq C_B(q) \), where \( C_B(q) \) is the endogenous bailout threshold corresponding to the switching policy. Define the function

\[
H_q = S^{social}_q - S_q
\]  

(A79)
By (15), $H_q$ satisfies the following system of ODEs:

$$L_1 H_q + (\lambda_1 - \lambda) C - (\lambda_1 - \lambda) \theta = 0 \text{ for } C \geq \theta$$
$$L_1 H_q = 0 \text{ for } q \leq C < \theta$$
$$L_2 H_q = 0 \text{ for } C_B(q) \leq C < q$$

(A80)

We also have $H_q \geq 0$ since the social cost is less than the private cost of equity ($\lambda \leq \lambda_1$). By arguments similar to those used in the proof of Proposition 1, $H_q$ has the following form:

$$H_q(C) = \begin{cases} 
W_1 C^{\gamma_1^+} + (\lambda_1 - \lambda) \frac{C}{r - \mu_1} - (\lambda_1 - \lambda) \frac{\theta}{r}, & \text{for } \theta \leq C, \\
W_2 C^{\gamma_1^-} + W_3 C^{-\mu_1}, & \text{for } q \leq C < \theta, \\
W_4 C^{\gamma_2^-} + W_5 C^{\gamma_2^+}, & \text{for } C_B(q) \leq C < q.
\end{cases}$$

(A81)

From the value matching condition $H_q(C_B(q)) = 0$ and that $H_q \geq 0$, we solve

$$W_4 = -WC_B^{-\gamma_2^-}, W_5 = WC_B^{-\gamma_2^+}, \text{ for some } W > 0$$

(A82)

An examination of the proof of Theorem 2 reveals that it suffices to show that

$$L_2 H_q |_{C=q^+} \leq 0$$

(A83)

to ensure that project 1 is always chosen when $C > q$. It is also straightforward to verify that (A83) is equivalent to

$$L_1 H_q |_{C=q^-} \geq 0.$$  

(A84)

Plugging (A82) into (A84) yields the equivalent condition

$$-(\gamma_2^- - \gamma_1^-)(\gamma_2^- - \gamma_1^-) \left( \frac{q}{C_B} \right)^{\gamma_2^-} + (\gamma_2^+ - \gamma_1^+)(\gamma_2^+ - \gamma_1^+) \left( \frac{q}{C_B(q)} \right)^{\gamma_2^+} \geq 0.$$  

(A85)

Since $\gamma_2^+ > \gamma_1^+ > \gamma_2^- > \gamma_1^-$, the left hand side of (A85) is always positive and thus the condition holds.

Part 2. We now show that $q^* \leq q_1^*$.

Consider a switching policy described by a trigger $q$, where $q \geq C_B(q)$, where $C_B(q)$ is the endogenous bailout threshold corresponding to the switching policy. Define the function

$$G_q = F_q^{social} - S_q^{social}$$

(A86)
By (15) and (17), $G$ satisfies the following system of ODEs:

\[
L_1G_q + \theta = 0 \text{ for } C > q \\
L_2G_q + \theta = 0 \text{ for } C_B(q) < C < q
\]

By arguments similar to those used in the proof of Proposition 1, $G$ has the following form for $C > q$:

\[
G_q(C) = Y_1 C^{\gamma^-} + \frac{\theta}{r}.
\]  \hspace{1cm} (A87)

Since $\frac{\theta}{r}$ is the value of $G_q(C)$ in the hypothetical scenario where debt is completely risk-free, we must have

\[
Y_1 < 0.
\]  \hspace{1cm} (A88)

Therefore,

\[
L_2(G_q) + \theta = \frac{1}{2} Y_1 \sigma G^2 (\gamma^-_1 - \gamma^-_2)(\gamma^-_1 - \gamma^+_2)C^{\gamma^-} < 0, \text{ if } C > q,
\]  \hspace{1cm} (A89)

because $Y_1 < 0$ and $\gamma^-_1 < \gamma^-_2$ by (A11).

By arguments analogous to the proof of Theorem 1, we have

\[
L_2S^{social}_{q^*_1} + (1 + \lambda_2)C - (1 + \lambda_2)\theta|_{C=q^*_1} + \leq 0
\]  \hspace{1cm} (A90)

An examination of the proof of Theorem 2 reveals that it suffices to show that

\[
L_2F^{social}_{q^*_1} + (1 + \lambda_2)C - \lambda_2\theta|_{C=q^*_1} + \leq 0
\]

to ensure that the regulator optimally chooses project 1 for all $C \geq q^*_1$, that is, the regulator’s optimal switching trigger is less than $q^*_1$. However, by (A86) and (A89),

\[
L_2F^{social}_{q^*_1} + (1 + \lambda_2)C - \lambda_2\theta|_{C=q^*_1} + = (L_2S^{social}_{q^*_1} + (1 + \lambda_2)C - (1 + \lambda_2)\theta \]
\[+ \quad L_2(G_q^*) + \theta)|_{C=q^*_1} < 0,
\]

where the inequality above follows from (A90) and (A89). Q.E.D.

\section*{A1.3. The Unregulated Bank’s Optimal Policy in the Dynamic Model with Debt Restructuring}

\textbf{Proof of Proposition 4}

The functional forms, (28) and (31), for the debt and equity values, respectively, follow using arguments identical to those used in Main Step 1 of the proofs of Theorem 1 and Proposition 1 in Section A1.1. The boundary condition, (29), at the upward restructuring threshold, $C_U$, arises from the fact that debt holders receive the par value of their bonds when debt is restructured
Proposition 5

at \( C_U \). To obtain the boundary condition, (30), for the debt value at bankruptcy, we make two observations. First, the linear homogeneity property of the model ensures that the total value of the restructured bank after bankruptcy equals \( kB \cdot [S(C_0) + D(C_0)] \) minus the new debt issuance cost, \( \varphi k_B D(C_0) \). Bankruptcy costs are proportional to the post bankruptcy value of the bank. Second, because the debt value must be continuous at \( C_B \), and ownership of the bank transfers to debt holders at bankruptcy, the debt value prior to bankruptcy equals the post bankruptcy bank value minus restructuring costs.

The boundary condition, (33), follows from the fact that the equity value is zero at bankruptcy. The boundary condition, (32), follows from two observations. First, by the linear homogeneity property, the total bank value after the upward restructuring equals \( \varphi_k C^0 \). Second, shareholders’ claim at \( C_U \) equals the total bank value after restructuring minus the required payment, \( D(C_U) = D(C_0) \) to debt holders. Q.E.D.

**Proof of Proposition 5**

The proof proceeds in several steps.

**Step 1.** An admissible policy for the bank is described by the sequence of vectors,

\[
\Gamma \equiv \left\{ (\tau_n, \theta^n, \xi^n, C^n_S, C^n_U, C^n_B) : n \geq 0 \right\},
\]

where \( \tau_0 = 0 \), \( \tau_n \) is an increasing sequence of \( \mathcal{F}_t \)-stopping times with \( \tau_n \to \infty \) a.s., and \( (\theta^n, \xi^n, C^n_S, C^n_U, C^n_B) \) is an \( \mathcal{F}_{\tau_n} \)-measurable random vector denoting the bank’s debt level, investment level, switching threshold, upward restructuring threshold, and bankruptcy threshold over the period, \( (\tau_n, \tau_{n+1}) \). Further, \( \tau_{n+1} = \min(\tau_{C^n_U}, \tau_{C^n_B}) \), where \( \tau_{C^n_U} = \inf \{ t \geq \tau_n : C_t > C^n_U \} \); \( \tau_{C^n_B} = \inf \{ t \geq \tau_n : C_t < C^n_B \} \) so that \( \tau_n \) are the times at which the bank either restructures its debt upwards or goes bankrupt, whichever occurs earlier. Finally, the banks’ gross-of-investment EBIT rate is continuous and satisfies \( \tilde{C}_{\tau^n_+} = \tilde{C}_{\tau^n_-} \). Therefore, \( C_t = (1 - K(\xi^n)) \tilde{C}_t \) for \( \tau_{n-1} \leq t < \tau_n \), and the EBIT rate evolves as in (2) and (3) with \( \xi = \xi^n \) over the interval \( (\tau_n, \tau_{n+1}) \).

**Step 2.** With dynamic restructurings, the “decision maker” changes over time; the shareholders after a restructuring differ from the shareholders before a restructuring. Consequently, we characterize the optimality of a policy recursively. We begin with some notation. If \( \Gamma \) is an admissible policy for the bank as described above, for each \( n > 0 \), let \( \Gamma^n \) be the restriction of the policy to the time period, \( [\tau_n, \infty) \), that is, the period after the \( n \)th restructuring. A policy \( \Gamma^* \equiv \{ (\tau^n_s, \theta^{n_s}, \xi^{n_s}, C^{n_s}_S, C^{n_s}_U, C^{n_s}_B) : n \geq 0 \} \) is a dynamically optimal policy for the bank if and only if it satisfies the following sequential optimality conditions for each \( n \geq 0 \).

First,

\[
(\xi^{n_s}, C^{n_s}_S, C^{n_s}_U, C^{n_s}_B) \equiv \arg \max_{(\xi^n, C^n_S, C^n_U, C^n_B)} S \left( \tau^n_s, (\theta^{n_s}, \xi^{n_s}, C^{n_s}_S, C^{n_s}_U, C^{n_s}_B), \Gamma^{n+1} \right),
\]
where $S \left( \tau_n^*; (\theta^n, \xi^n, C^n_S, C^n_U, C^n_B), \Gamma^{n+1*} \right)$ denotes the shareholder value at stopping time $\tau_n^*$ when the future policy variables are given by $(\theta^n, \xi^n, C^n_S, C^n_U, C^n_B)$ during the $n$-th restructuring interval and by the policy $\Gamma^{n+1*}$ in future restructuring intervals.

Second,

$$\theta^n* = \arg \max_{\theta} F \left( \tau_n^*; (\theta, \xi^n*, C^n_S, C^n_U, C^n_B), \Gamma^{n+1*} \right),$$  \hspace{1cm} (A92)

where $F \left( \tau_n^*; (\theta, \xi^n*, C^n_S, C^n_U, C^n_B), \Gamma^{n+1*} \right)$ is the total bank value at stopping time $\tau_n^*$ with the arguments interpreted as above. Note that the conditions, (A91) and (A92) must hold both “on” and “off” the equilibrium path.

**Step 3.** We now show that, for any admissible policy $\Gamma$, the initial total bank value satisfies

$$F(C_0; \Gamma) \leq c \hat{C}_0,$$ \hspace{1cm} (A93)

for some constant $c$. Indeed, given any debt level $\theta$, the combined cash flows to debt and equity holders at any time $t$ cannot exceed $C_t$ because

$$(1 - \lambda_1)(C_t - \theta) + \theta = (1 - \lambda_1)C_t + \lambda_1\theta \leq C_t, \quad \text{if } C_t \geq \theta,$$

$$(1 + \lambda_2)(C_t - \theta) + \theta = (1 + \lambda_2)C_t - \lambda_2\theta \leq C_t, \quad \text{if } C_t < \theta.$$

Therefore,

$$F(C_0; \Gamma) \leq \int_0^{\infty} e^{-rt} C_t(\Gamma) dt \leq \int_0^{\infty} e^{-rt} \hat{C}_t(\Gamma) dt \leq \frac{1}{r - \mu_1} \hat{C}_0,$$

where the last inequality follows from the fact that project 1 has a higher drift and NPV and thus is chosen all the time if there is no possibility of default. For notational convenience (with a slight abuse of notation), we denote below debt and equity values in terms of the gross-of-investment cash flows, i.e., $D = D(\hat{C})$, $S = S(\hat{C})$. The bound on total bank value, (A93), then implies

$$D(\hat{C}_0; \Gamma) + S(\hat{C}_0; \Gamma) \leq c \hat{C}_0.$$  \hspace{1cm} (A94)

**Step 4.** We use a fixed point argument to show the existence of a dynamically optimal restructuring policy that satisfies the scaling properties described in Section 4.1.

Consider the subclass of policies where the bank can restructure at most $n$ times where $n \geq 1$. For $n = 0$, this is simply the basic model of Section 2 that cannot restructure its debt. Denote the optimal debt and equity values of the bank within this subclass by $D^n$ and $C^n$, respectively. The value functions exist by (A94). Let us make the inductive assumption that $D^n$ and $C^n$ are linearly homogeneous in the earnings level; the assumption is satisfied for $n = 0$ by the results of Section 3. Now consider the problem where the bank can restructure at most once, but the continuation debt and equity values after a restructuring are $D^n$ and $C^n$, respectively. For a given restructuring policy, $\pi = (\theta, \xi, C_S, C_U, C_B)$, for the bank that restructures at most
once, it is easy to see that the debt and equity values are given by Proposition 4 except that the boundary conditions are replaced by the following conditions:

\[
\begin{align*}
D^{n+1}(\hat{C}_U; \pi) &= D^n(\hat{C}_0), \\
S^{n+1}(\hat{C}_U; \pi) &= k_U(D^n(\hat{C}_0) + S^n(\hat{C}_0) - \varphi D^n(\hat{C}_0)), \\
D^{n+1}(\hat{C}_B; \pi) &= (1 - \alpha)k_B(D^n(\hat{C}_0) + S^n(\hat{C}_0) - \varphi D^n(\hat{C}_0)), \\
S^{n+1}(\hat{C}_B; \pi) &= 0.
\end{align*}
\] (A95)

Note that, in the above, we use the linear homogeneity of the continuation debt and equity values. The optimal policy variables then satisfy

\[
(\xi^{n+1}, C_S^{n+1}, C_U^{n+1}, C_B^{n+1}) \equiv \arg \max_{(\xi, C_S, C_U, C_B)} S^{n+1}(\hat{C}_0; \pi); \quad \theta^{n+1} \equiv \arg \max_{\theta} F(\hat{C}_0; (\theta, \xi^{n+1}, C_S^{n+1}, C_U^{n+1}, C_B^{n+1})).
\] (A96)

The optimal policy variables exist using standard continuity and compactness arguments that we omit for brevity. Further, it follows immediately from the equations that determine \(D^{n+1}\) and \(S^{n+1}\) that they are also linearly homogeneous in the earnings level.

Let

\[
D(\hat{C}_0) = \lim_{n \to \infty} D^n(\hat{C}_0); S(\hat{C}_0) = \lim_{n \to \infty} S^n(\hat{C}_0).
\]

The limiting value functions defined above exist by (A94). It is easy to show that the policy vector, \((\theta^n, \xi^n, C_S^n, C_U^n, C_B^n)\) is bounded for all \(n\). Consequently, there exists a subsequence that converges to the limit vector, \((\theta^*, \xi^*, C_S^*, C_U^*, C_B^*)\). By the optimality of the policies \((\theta^n, \xi^n, C_S^n, C_U^n, C_B^n)\) within the subclass of policies characterized by at most \(n\) restructurings, it follows that \((\theta^*, \xi^*, C_S^*, C_U^*, C_B^*)\) determines the sequentially optimal policy for the original problem. Further, \(D\) and \(S\) are the corresponding optimal debt and equity value functions.

**Step 5.** Consider the optimal restructuring policy described as in the discussion before the statement of the proposition by \((\theta^*, \xi^*, C_S^*, C_B^*, C_U^*)\). By local perturbation arguments similar to those used in the proofs of Theorems 1 and 2, we obtain that

\[
L_2 S + (1_{C \geq \theta^*}(1 - \lambda_1)(C - \theta^*) + 1_{C < \theta^*}(1 + \lambda_2)(C - \theta^*))|_{C^+} \leq 0.
\] (A97)

and

\[
L_1 S + (1_{C \geq \theta^*}(1 - \lambda_1)(C - \theta^*) + 1_{C < \theta^*}(1 + \lambda_2)(C - \theta^*))|_{C^-} \leq 0.
\] (A98)

We also have from Ito’s lemma that

\[
\begin{align*}
L_1 S + (1_{C \geq \theta^*}(1 - \lambda_1)(C - \theta^*) + 1_{C < \theta^*}(1 + \lambda_2)(C - \theta^*)) &= 0 \text{ for } C > C_S^* \\
L_2 S + (1_{C \geq \theta^*}(1 - \lambda_1)(C - \theta^*) + 1_{C < \theta^*}(1 + \lambda_2)(C - \theta^*)) &= 0 \text{ for } C < C_S^*
\end{align*}
\] (A99)
The super contact condition, (34) follows immediately from (A97), (A98), and (A99). The smooth pasting conditions, (35) and (36), follow from the standard local perturbation arguments applied to the boundary conditions (33) and (32). Q.E.D.

A2. Excess Costs of Equity Relative to Bank Debt

In this section, we formalize some of the arguments discussed in Section 2 for the existence of excess costs of bank equity relative to bank debt. The discussion here provides a simple formalization of the argument that (i) investors’ preference for liquid debt securities (especially demand deposits); (ii) banks’ unique role in providing such securities; and (iii) the relative scarcity of bank capital stemming from the fact that banks face a “limited pledgeability” constraint a la Holmstrom and Tirole (1997, 2011); together generate a premium for bank debt relative to bank equity. For simplicity, we consider a one-period economy with a continuum of identical investors of mass one each endowed with one unit of capital, and a continuum of banks with mass \( k \ll 1 \) each endowed with one unit of capital that could be interpreted as the “internal” capital of banks. As in Holmstrom and Tirole (2011), banks face a “limited pledgeability” constraint that limits the amount of external capital that banks can raise to a proportion \( \theta \in R_+ \) of their internal capital. As argued by Holmstrom and Tirole (among others), the constraint arises from the fact that banks must have a minimum “skin in the game” to deter moral hazard. For example, banks could have alternate value-destroying investment opportunities that provide them with non-transferrable private benefits. The incentive constraint that prevents such moral hazard limits the amount of income that banks can pledge to outside investors to a proportion of their internal capital. Consequently, the maximum amount of external capital that each bank can raise is

\[
K = \theta k, \tag{A100}
\]

where \( K \ll 1 \), that is, the total external capital that banks can raise is much smaller than the available capital held by investors.

Without loss of generality, let us focus on a representative investor and a representative bank. The available investment opportunities in the economy comprise of “bank debt” that provides a return of \( R_D(\omega) \) per unit of capital (we allow the return to be random to allow for the possibility of bank default) and \( N \) other securities/investment opportunities that provide returns of \( R_1(\omega), R_2(\omega), ..., R_N(\omega) \) per unit of capital, where \( \omega \) denotes an element of the underlying probability space. Bank equity is a “redundant” security in that its payoff can be replicated by a combination of the payoffs of other securities in the economy. Bank debt, however, is unique and cannot be replicated by other securities. The representative investor has preferences for date 1 consumption that are described by the utility function \( U(\cdot) \), which satisfies the usual Inada conditions to ensure interior solutions.

First, consider the problem of the social planner who optimally chooses the amount of capital
to allocate to each of the securities. The planner solves

$$\max_{k_D, k_E, k_F} E \left[ U(k_D R_D(\omega) + \sum_{i=1}^{N} k_i R_i(\omega)) \right]$$  \hfill (A101)

such that

$$k_D + \sum_{i=1}^{N} k_i \leq 1;$$

$$k_D \leq K.$$  \hfill (A102)

The first constraint above is the aggregate constraint on the total amount of capital, while the second is the constraint on the amount of bank debt. There are, in general, constraints on the amounts of capital that can be allocated to the other investment opportunities, but we assume that the constraint on bank capital is the binding constraint in equilibrium, which captures the relative scarcity of bank capital.

If $\lambda$ and $\mu$ are the strictly positive Lagrange multipliers corresponding to the binding constraints (A102), then the first order conditions of the optimization program (A101) are

$$E \left[ U'(k_D^* R_D(\omega) + \sum_{i=1}^{N} k_i^* R_i(\omega)) R_D(\omega) \right] = \lambda + \mu$$  \hfill (A103)

$$E \left[ U'(k_D^* R_D(\omega) + \sum_{i=1}^{N} k_i^* R_i(\omega)) R_j(\omega) \right] = \lambda; \ j = 1, ..., N$$  \hfill (A104)

Now consider the investor’s problem. Let $p_D, \{p_i; i = 1, ..., N\}$ be the prices of the bank’s debt and the $N$ other securities. The investor has an initial amount of wealth 1 and chooses her portfolio to maximize her expected utility, that is, she solves

$$\max_{x_D, x_E, x_F} E \left[ U(x_D R_D(\omega) + \sum_{i=1}^{N} x_i R_i(\omega)) \right]$$  \hfill (A105)

such that

$$x_D p_D + \sum_{i=1}^{N} x_i p_i \leq 1,$$

where $x_D, \{x_i; i = 1, ..., N\}$ are the quantities of bank debt and the other $N$ securities, respectively, purchased by the investor. In particular, the individual investor does not internalize the aggregate constraint on bank capital. The first-order conditions of the investor’s optimization
program then imply that (normalizing the Lagrange multiplier to one)

\[ p_D = E \left[ U' \left( k_D^* R_D(\omega) + \sum_{i=1}^{N} k_i^* R_i(\omega) \right) R_D(\omega) \right], \]
\[ p_j = E \left[ U' \left( k_D^* R_D(\omega) + \sum_{i=1}^{N} k_i^* R_i(\omega) \right) R_j(\omega) \right], \]

(A106)

where we have used the fact that, at the optimum, the representative investor purchases all the
securities and the solutions to the planning and investor’s problems coincide. From (A103) and
(A106),

\[ p_D > p_j; \ j = 1, \ldots, N. \]

From the above, we easily see that, for each unit of capital raised, the return on bank debt is
lower than that of other securities including bank equity.

**A3. The Estimation Procedure**

As we discussed in Section 5, the eight structural parameters that we need to estimate are

\[ \Lambda = \{ \mu_1, 0, \sigma_1, \mu_2, 0, \sigma_2, \lambda, \lambda_2, \kappa, \psi \}. \]

We use the method of simulated moments to obtain the baseline values of the parameters.
For a given candidate set of parameters \( \Lambda \), we simulate the model to create many artificial
panels of banks. We then use the simulated panels to compute “simulated moments” that are
counterparts to the empirical moments in the real data that are listed in Panel C of Table 1.
We calculate these moments for each artificial panel and then compute their averages across the
panels. We denote by \( V(\Lambda) \) the vector of these average simulated moments, and \( \hat{V} \) the vector
of the empirical values of these moments. The baseline set of parameter values, \( \Lambda^* \), solves the
following optimization problem:

\[ \Lambda^* = \arg \min_{\Lambda} (V(\Lambda) - \hat{V})^T W (V(\Lambda) - \hat{V}), \]

(A107)

where \( W \) is the diagonal, positive definite matrix with entries equal to the inverse of the empirical
moments given by \( \hat{V} \). In other words, we minimize the distance between simulated and empirical
moments.

We now describe how we simulate the model to compute the simulated moments.

(i) First, for each candidate vector of parameter values, \( \Lambda \), we compute the optimal policy
variables as discussed in Section 4.4. Let \( \Pi \) denote the vector of policy variables.

(ii) Second, given the vector, \( \Pi \), of policy variables, we simulate the model. Because the empirical
moments are obtained from actual or physical data, we need to simulate the model under the
physical probability measure. By (2), (3) and Girsanov’s theorem, the volatilities of the banks’ EBIT rate are the same in the physical and risk-neutral measures. The physical drift, $\bar{\mu}_t(\xi)$, of the representative bank’s EBIT rate at date $t$ is related to the risk-neutral drift $\mu_t(\xi)$ as $\bar{\mu}_t(\xi) = \mu_t(\xi) + \varsigma_1\sigma_t$, where $\varsigma$ is the market price of risk of banks and $\sigma_t$ is the volatility of the project chosen at time $t$. Here, $\xi$ is the bank’s optimal investment computed in Step (i) above. Hence, the bank’s EBIT rate evolves as follows under the physical measure.

$$dC_t(\xi) = (\bar{\mu}_{1,0} + \xi) C_t(\xi) dt + \sigma_1 C_t(\xi) dB^1_t = \bar{\mu}_1(\xi) C_t(\xi) dt + \sigma_1 C_t(\xi) dB^1_t,$$

$$dC_t(\xi) = (\bar{\mu}_{2,0} + \xi) C_t(\xi) dt + \sigma_2 C_t(\xi) dB^2_t = \bar{\mu}_2(\xi) C_t(\xi) dt + \sigma_2 C_t(\xi) dB^2_t,$$

where $B^1, B^2$ are Brownian motions. We use the CAPM model to estimate the risk premium of banks to be 3.54% over the period 1991–2008 and use it as the risk premium, $\varsigma_1\sigma_1$ of the low-risk project. Consequently, we set the market price of risk, $\varsigma = 3.54%/\sigma_1$, and compute the physical drifts $\bar{\mu}_i(\xi) = (\mu_{i,0} + \xi) + \varsigma\sigma_i$ accordingly.

We discretize the earnings process of a bank $C^i_t$ as follows:

$$C^i_t = C^i_{t-\Delta t}e^{(\bar{\mu}_t(\xi)-\frac{\varsigma^2}{2})\Delta t+\varsigma_1\sqrt{\Delta t}\omega^i},$$  \hspace{1cm} (A108)

where the length of a period $\Delta t$ is set to be one month. In the above, $(\bar{\mu}_t(\xi), \sigma_t) = (\bar{\mu}_1(\xi), \sigma_1)$ if the low-risk project is chosen, and $(\bar{\mu}_t(\xi), \sigma_t) = (\bar{\mu}_2(\xi), \sigma_2)$ if the high-risk project is chosen.

As mentioned in Section A3, we incorporate the fact that the shocks faced by banks have systematic and idiosyncratic components. Accordingly, we decompose the random shock variable $w^i_t$ in (A108) as a linear combination of a systematic and idiosyncratic component. For the $i$-th bank, the shock is given by

$$w^i_t = \rho z^{sys}_t + \sqrt{1-\rho^2} z^i_t,$$

where $z^{sys}_t$ is the systematic shock and $z^i_t$ is the idiosyncratic shock to bank $i$. We use $\rho = 0.378$, which is the average correlation of bank stock returns with the banking sector returns in our empirical sample. We draw $z^{sys}_t$ and $z^i_t$ from independent unit normal distributions. For each realized systematic shock path $\{z^{sys}_t\}_{t=1,...,T}$, we consider $N$ idiosyncratic shock paths $\{z^i_t\}_{t=1,...,T,i=1,...,N}$. We choose $N = 680$ in our simulation because this equals the average number of banks each year in our empirical data. We also consider 100 different systematic shock histories. Therefore, we create 100 artificial panels of data with each panel corresponding to one path of the systematic shocks. In each panel, there are 680 banks, corresponding to the 680 idiosyncratic risk paths.

We simulate the random evolution path of each bank for 50 years (600 months). To minimize the impact of initial conditions, we use only the panel data consisting of the sample paths from the 31st year to the 50th year (20 years or 240 months) to compute the simulated moments. We calculate the simulated moments for each artificial panel, and then calculate the average of the
moments across all of the 100 panels.

We use the following parametric bootstrapping procedure to determine the standard errors of the estimated parameters and simulated moments. First, we create 100 new panel datasets of paths of random shocks, $\Upsilon_i$, $1 \leq i \leq 100$, by redrawing the systematic and idiosyncratic shocks as described above. Second, for any candidate set of parameters $\Lambda$, we compute $V(\Lambda, \Upsilon_i)$, the vector of simulated moments calculated based on the parameter vector, $\Lambda$, and the paths of random shocks $\Upsilon_i$. We solve the following optimization problem based on the dataset $\Upsilon_i$ to obtain a reoptimized set of parameters, $\Lambda_i^*$:

$$\Lambda_i^* = \arg \min_{\Lambda} (V(\Lambda, \Upsilon_i) - \tilde{V})^T W (V(\Lambda, \Upsilon_i) - \tilde{V}).$$

(A109)

The set $\{\Lambda_i^* : 1 \leq i \leq 100\}$ is the set of bootstrapped parameter estimates from which we compute the standard deviations/errors of the baseline parameters, $\Lambda^*$. We compute the standard deviations of the moments from the set of bootstrapped moments $V(\Lambda_i^*, \Upsilon_i)$.

A4. Intermediate Model: Target Capital Requirement and Recapitalization

We describe the model and the associated optimization problems in the intermediate scenario in which the regulator imposes a target capital requirement and recapitalizes the bank, while the bank chooses the investment, risk-taking and upward restructuring policies.

As in the other cases considered in Section 4, we maintain the sequential game structure in which the regulator moves first by setting the debt level. The bank then chooses its investment, risk-taking and upward restructuring policies. Finally, the regulator chooses the recapitalization threshold. The strategies of the players are sequentially or dynamically optimal. The combined policy of the bank and the regulator before the first restructuring time is described by a vector $(\theta^{\text{reg}}, \xi, C_U, C_S, C_{\text{reg}}^D)$, where $(\xi, C_U, C_S)$ determines the bank’s policy, and $(\theta^{\text{reg}}, C_{\text{reg}}^D)$ is the regulator’s policy. If $C_t$ first reaches $C_U$ before reaching $C_{\text{reg}}^D$, then after the upward restructuring, the regulator determines the new debt level, $\theta^{\text{reg}}_1$; the bank then determines the new investment, upward restructuring and switching policies by a vector $(\xi_1, C_{U1}, C_{S1})$; and the regulator determines the recapitalization threshold $C_{\text{reg}}^{D1}$. The new problem facing the bank and the regulator is a “rescaled” version of the problem facing them at time zero where the initial cash flow level in the new problem is $C_U$ instead of $C_0$. Again, because investment is scale-invariant, the new (bank’s and regulator’s) policy $(\theta^{\text{reg}}_1, \xi_1, C_{U1}, C_{S1}, C_{\text{reg}}^{D1}) = (k_U \theta^{\text{reg}}, \xi, k_U C_U, k_U C_S, k_U C_{\text{reg}}^D)$ would be optimal if $(\theta^{\text{reg}}, \xi, C_U, C_S, C_{\text{reg}}^D)$ is the optimal policy before the first restructuring. Similar scaling properties of the optimal policy hold at the recapitalization threshold $C_{\text{reg}}^{D1}$ if it is reached first. Therefore, a dynamically optimal policy in the intermediate model is determined completely by the policy vector $(\theta^{\text{reg}}, \xi, C_U, C_S, C_{\text{reg}}^D)$ before the first restructuring time. The debt, equity, and social values of the bank are described in the same forms as in Propositions 4 and 6.
We now describe the determination of the optimal policy variables in this intermediate model that is described by the vector, \((\theta^{\text{int}}, \xi^{\text{int}}, C_{U}^{\text{int}}, C_{S}^{\text{int}}, C_{D}^{\text{int}})\), where \(\theta^{\text{int}}\) and \(C_{D}^{\text{int}}\) are the regulator’s optimal choice of the debt level and recapitalization threshold.

(i) Suppose that the debt level choice of the regulator and the bank’s investment, upward restructuring and switching policy are described by the vector, \((\theta^{\text{reg}}, \xi, C_{U}, C_{S})\). The regulator’s choice of the recapitalization threshold, \(C_{D}^{\text{reg}}(\theta^{\text{reg}}, \xi, C_{U}, C_{S})\), maximizes the bank’s social value, that is,

\[
C_{D}^{\text{reg}}(\theta^{\text{reg}}, \xi, C_{U}, C_{S}) = \arg\max_{C_{D}^{\text{reg}}} F_{0}^{\text{social}}(\theta^{\text{reg}}, \xi, C_{U}, C_{S}, C_{D}^{\text{reg}}). \tag{A110}
\]

Consistent with the Stackelberg structure, the regulator optimally chooses the recapitalization threshold in response to the other policy variables, \((\theta^{\text{reg}}, \xi, C_{U}, C_{S})\) chosen by the bank and the regulator.

(ii) Given the regulator’s policy function, \(C_{D}^{\text{reg}}(\theta^{\text{reg}}, \xi, C_{U}, C_{S})\), which solves (A110), and the debt level \(\theta^{\text{reg}}\), the bank’s choices, \((\xi^{\ast}(\theta^{\text{reg}}), C_{U}^{\ast}(\theta^{\text{reg}}), C_{S}^{\ast}(\theta^{\text{reg}}))\), maximize its equity value, that is,

\[
(\xi^{\ast}(\theta^{\text{reg}}), C_{U}^{\ast}(\theta^{\text{reg}}), C_{S}^{\ast}(\theta^{\text{reg}})) = \arg\max_{(\xi, C_{U}, C_{S})} S_{0}(\theta^{\text{reg}}, \xi, C_{U}, C_{S}, C_{D}^{\text{reg}}(\theta^{\text{reg}}, \xi, C_{U}, C_{S})). \tag{A111}
\]

Consistent with the leader-follower structure, the bank makes its choices rationally anticipating the regulator’s best response to its choices.

(iii) Finally, the regulator’s choice of debt level, \(\theta^{\text{int}}\), maximizes the bank’s social value at date 0.

\[
\theta^{\text{int}} = \arg\max_{\theta^{\text{reg}}} F_{0}^{\text{social}}(\theta^{\text{reg}}, \xi^{\ast}(\theta^{\text{reg}}), C_{U}^{\ast}(\theta^{\text{reg}}), C_{S}^{\ast}(\theta^{\text{reg}}), C_{D}^{\text{reg}}(\theta^{\text{reg}}, \xi^{\ast}(\theta^{\text{reg}}), C_{U}^{\ast}(\theta^{\text{reg}}), C_{S}^{\ast}(\theta^{\text{reg}}))). \tag{A112}
\]

The other optimal policy thresholds are given by \(\xi^{\text{int}} = \xi^{\ast}(\theta^{\text{int}})\), \(C_{U}^{\text{int}} = C_{U}^{\ast}(\theta^{\text{int}})\), \(C_{S}^{\text{int}} = C_{S}^{\ast}(\theta^{\text{int}})\), and \(C_{D}^{\text{int}} = C_{D}^{\text{reg}}(\theta^{\text{int}}, \xi^{\text{int}}, C_{U}^{\text{int}}, C_{S}^{\text{int}})\).

**A5. Alternate Model for Estimation: Random Recapitalization**

We briefly describe the alternate model where the regulator recapitalizes the bank with some probability when it is sufficiently distressed. The model is similar to the model in Section 4.4, but with two key changes. First, the regulator recapitalizes the bank with some probability \(\phi \in (0, 1)\) if it is sufficiently distressed, where the probability \(\phi\) is known to all market participants. If the regulator does not recapitalize the bank, it goes bankrupt when its equity value falls to zero. Second, there is no debt dilution if the bank is recapitalized. As in Section 4.4, the stake of existing shareholders if the bank is recapitalized equals their “outside option”, which is the value of equity if the regulator chooses not to intervene at all and bail out the bank. Suppose that a restructuring policy of the hybrid bank is described by the 5-tuple, \((\theta, C_{U}, C_{S}, C_{D}^{\text{reg}}, C_{B})\), where
the recapitalization threshold is $C_D^{reg}$ if the regulator recapitalizes the bank and the bankruptcy threshold is $C_B$ if the regulator does not recapitalize the bank. The following proposition characterizes the bank’s equity, debt and social values before the first restructuring, for a given policy, $(\theta, C_U, C_S, C_D^{reg}, C_B)$.

**Proposition A9.** Suppose that the regulator first recapitalizes the bank at the threshold, $C_D^{reg}$, with some probability $\phi \in (0, 1)$, where $C_D^{reg} \leq \theta \leq C_S \leq C_U$. Let $k_U = C_U/C_0$ and $k_D = C_D^{reg}/C_0$.

(i) The bank’s debt value at any date $t$ prior to the first (upward or downward) restructuring is differentiable for $C_t \in (C_D^{reg}, C_U)$, and is determined by

$$D(C_t) = \begin{cases} B_1 C_t^{\gamma_1} + B_2 C_t^{\gamma_2} + \frac{\theta}{r}, & C_S < C_t \leq C_U \\ B_3 C_t^{\gamma_1} + B_4 C_t^{\gamma_2} + \frac{\theta}{r}, & C_D^{reg} < C_t \leq C_S \end{cases}$$

(A113)

along with the boundary conditions

$$D(C_U) = D(C_0), \quad D(C_D^{reg}) = \phi D(C_0) + (1 - \phi) D_{unreg}(C_D^{reg})$$

(A114)

where $D_{unreg}(C_D^{reg})$ is the debt value of the unregulated bank at the threshold $C_D^{reg}$ that is given by part (i) of Proposition 4.

(ii) The equity value of the bank at any date $t$ prior to the first restructuring is differentiable for $C_t \in (C_D^{reg}, C_U)$, and is determined by

$$S(C_t) = \begin{cases} A_1 C_t^{\gamma_1} + A_2 C_t^{\gamma_2} + \frac{(1-\lambda_1)C_t}{r-\mu_1} - \frac{(1-\lambda_1)\theta}{r}, & C_S < C_t \leq C_U \\ A_3 C_t^{\gamma_1} + A_4 C_t^{\gamma_2} + \frac{(1-\lambda_1)C_t}{r-\mu_2} - \frac{(1-\lambda_1)\theta}{r}, & \theta < C_t \leq C_S \\ A_5 C_t^{\gamma_1} + A_6 C_t^{\gamma_2} + \frac{(1+\lambda_2)C_t}{r+\mu_2} - \frac{(1+\lambda_2)\theta}{r}, & C_D^{reg} < C_t \leq \theta \end{cases}$$

(A115)

$$S(C_D^{reg}) = S_{unreg}(C_D^{reg}), \quad S(C_U) = k_U S(C_0) + (1 - \varphi) k_D D(C_0) - D(C_0)$$

(A116)

In (A116), $S_{unreg}(C_D^{reg})$ is the equity value of the unregulated bank at the threshold, $C_D^{reg}$, that is given by part (i) of Proposition 4.

(iii) The bank’s social value at any date $t$ prior to the first restructuring is differentiable for $C_t \in (C_D^{reg}, C_U)$, and is determined by

$$F_{social}(C_t) = \begin{cases} \frac{(1-\lambda_1)C_t}{r-\mu_1} + \frac{\lambda\theta}{r} + Y_1 C_t^{\gamma_1} + Y_2 C_t^{\gamma_2}, & C_S \leq C_t \leq C_U, \\ \frac{(1-\lambda_1)C_t}{r-\mu_2} + \frac{\lambda\theta}{r} + Y_3 C_t^{\gamma_1} + Y_4 C_t^{\gamma_2}, & \theta \leq C_t < C_S \\ \frac{(1+\lambda_2)C_t}{r+\mu_2} - \frac{\lambda\theta}{r} + Y_5 C_t^{\gamma_1} + Y_6 C_t^{\gamma_2}, & C_D^{reg} \leq C_t < \theta, \end{cases}$$

(A117)
\[ F_{social}(C_{\text{reg}}) = \phi(1 - \omega)[k_D F_{social}(C_0) - \varphi k_D D(C_0)] + (1 - \phi) F_{social,unreg}(C_{D}^{\text{reg}}) \]  
(A118)

\[ F_{social}(C_U) = k_U F_{social}(C_0) - \varphi k_U D(C_0), \]  
(A119)

where \(F_{social,unreg}(C_{D}^{\text{reg}})\) is the social value of the unregulated bank at the threshold, \(C_{D}^{\text{reg}}\).

The optimal policies of the bank and the regulator are determined as in Section 4.4.