Timing Ventures:
The Underinvestment Problem

By

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ABSTRACT

The incentives to exercise growth options, referred to here as timing ventures, are distorted by the presence of a prior claim to venture value. This distortion is the underinvestment problem of Myers (1977). We characterize the solution to the optimal exercise policy. The solution is a stopping time that does two things: (i) it indicates the time to undertake the venture when it is optimal to do so, or (ii) it indicates when the venture finishes out of the money in the sense of having zero time value. This solution depends on whose value is being maximized. If there is a prior claim to venture value, the solution that maximizes the value of the residual claim has the exercise strategy that is later than the solution that maximizes the value of the venture. However, this venture finishes out of the money before the value maximizing time.
1 Introduction

The flexibility to time the irreversible investment in plant and equipment provides the motivation for viewing such investment decisions as the exercise of an American call option. The recognition of flexibility in the timing, scale, and/or the operation of an investment has spawned the real options approach to investment under uncertainty. The literature on this approach is now vast.\(^1\)

Mauer and Ott (2000, p. 153) review the literature that examines the interaction of financing and investment decisions in the context of real options. Recently, Grenadier and Wang (2003) have examined real options in the principal-agent setting where a manager is delegated the responsibility of deciding when to exercise an investment option. In the presence of both moral hazard and adverse selection, they find that the manager’s decisions “. . . differ significantly from that of the first-best no-agency solution.”\(^2\)

One objective of this paper is point out that the incentive to exercise such investment options, what we refer to here as timing ventures, may be distorted even when the owner of the option does not delegate the exercise decision or when the owner delegates the decision to a manager who acts in the interest of the owner. The distortion comes from the presence of a prior claim to venture value in the firm that owns the venture. In the presence of a prior claim, owners would always decide to undertake the venture later than the value maximizing strategy.

This distortion of investment incentives is the well known problem of debt overhang. Myers (1977), in a seminal contribution to corporate finance, showed that in the presence of a prior claim to value, a debt claim, owners of a value increasing investment opportunity may choose to forego the opportunity if they have to contribute
the capital to undertake it. Myers coined the terms “growth option” and “real option” for such investment opportunities and the distortion in investment incentives has come to be known as the underinvestment problem. In the real options context, growth opportunities are American type options. It is not a far stretch to think that the underinvestment problem of Myers is an instance of the postponement of a growth option beyond its optimal exercise in the sense of value maximization.

This phenomenon is demonstrated here in a fairly general discrete time setting under a weak assumption on the value of the prior claim. From the perspective of value maximization, the timing venture is an optimal stopping problem that has a well characterized solution. The solution is a stopping time that does two things: (i) it indicates the time to exercise the American call, to undertake the venture, when it is optimal to do so, or (ii) it indicates when the option finishes out of the money in the sense of having zero time value.

This solution depends on whose value is being maximized. If the venture is all equity financed, the solution is the one that maximizes the value of the venture. If there is a prior claim to venture value, the solution that maximizes the value of the residual claim, the owners’ claim, has the exercise strategy that is later than the solution that maximizes the value of the venture. However, this venture finishes out of the money before the value maximizing time.

Myers showed that at the value maximizing time, the incentive of the residual claimant was to “… in some states of nature, pass up valuable investment opportunities.”^3 He did not characterize the optimal strategy of the residual claimant, however, leaving open the question of when the residual claimant exercises the option or when the option
of the residual claimant finishes out of the money. The other objective of this paper is to answer this question.

It turns out that passing up a valuable investment opportunity may come well before the value maximizing time. The intuition for this result is straightforward. The presence of a prior claim raises the exercise price of the residual claimant’s call option. As a result the option may finish out of the money before the value maximizing time. In those cases, however, it is possible to simply view the owner as waiting forever to exercise the option.

If, however, the residual claim in the levered venture actually undertakes the venture (exercises the option), then the time at which this takes place is later than the value maximizing time, i.e., the unlevered venture would have been undertaken earlier. The intuition for this result comes from the fact that owners’ interest in a levered venture is a non-decreasing convex transformation of venture value. Thus owners’ interests in a levered venture are risk loving compared to owners’ interests in an unlevered venture. The result that provides the intuition, established in Nachman (1975), states that the more risk loving are the interests of the value maximizer the more risk will be taken in the form of a longer optimal exercise strategy. This result applies here as well.

So owners of a levered venture postpone exercising their growth option beyond the value maximizing exercise time. As a result, they invest suboptimally relative to owners in an unlevered venture. The problem gets worse the greater the leverage. Not only are levered owners risk loving compared to unlevered owners, the greater the leverage the more risk loving they become and hence the longer their optimal exercise strategy.
The results here clarify, at least in the discrete time setting, the character of the underinvestment problem. In a geometric Brownian motion model of output price Mauer and Ott (2000) solve for the optimal capital structure and investment policy of a firm that has a growth option to expand the scale of operations. In numerical simulations they show, consistent with the results here, that the optimal exercise price that triggers exercise of the growth option is higher for the levered equity than for unlevered equity, which is higher than the trigger price for the optimal capital structure.

In this paper we take the capital structure of the firm, the presence of the prior claim to venture value, as given. The model is presented in Section 2. Optimal timing strategies are characterized in Section 3. The optimal strategies of levered owners and unlevered owners are compared in Section 4. The effect of an increase in leverage on these optimal exercise strategies is derived in Section 5. Section 6 concludes the paper. Proofs and technical results are gathered in the Appendix.

2 The Model

A firm has a timing venture. Time is in discrete dates \( t = 0,1,\ldots,T \), with horizon \( T \), the expiration of the venture. Now is \( t = 0 \). The venture can be undertaken (the growth option can be exercised) at any date \( t = 0,1,\ldots,T \). If the venture is undertaken at date \( t \), initial capital \( K_t \) (exercise price) is required to obtain the venture worth \( Y_t \). The value \( Y_t \) is the change in the market value of the firm that owns the venture and \( Y_t - K_t \) is the net present value.

The prior claim to venture value is \( D_t \) if the venture is undertaken at date \( t \). The value \( D_t \) includes any explicit payments due the prior claimant and the change in market
value of the prior claim. The change in market value of the residual claim at date \( t \) is therefore \( Y_t - K_t - D_t \) if the venture is undertaken at date \( t \).

Save for the weak regularity assumption we make in Section 4 below, the nature of the prior claim is not important. Typically this would be a creditor’s claim. It may however be a tax claim by some government. There is an interesting case developed in the Appendix where this could be a fractional share claim, say of a venture capitalist who may have contributed capital to obtain the right to the venture but cannot be forced to contribute a fractional share of the cost of undertaking the venture. Whatever the nature of the prior claim, it constitutes leverage in the venture, hence our terminology regarding the venture with no prior claim as unlevered and regarding the venture with a prior claim as levered.

The values \( Y_t, K_t, D_t \) are random variables whose values are known by date \( t \). We assume the same economic circumstances as Myers (1977, Section 4.1) for the existence of these values. In fact, we are in the situation of the discrete choice problem described in Myers (1977, Figure 4, p. 166). We let \( X_t = (Y_t - K_t)^+ \) and \( \bar{X}_t = (Y_t - K_t - D_t)^+ \) to denote interests of limited liability residual claims in the unlevered and levered venture, respectively.

These and all other random variables are defined on a probability space \((\Omega, \mathcal{F}, P)\), and are adapted to the sequence \( \{\mathcal{F}_t\} \) of \( s \)-algebras, where \( \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{T-1} \subseteq \mathcal{F}_T = \mathcal{F} \). \( \mathcal{F}_t \) are the events knowable by date \( t \), the firm’s information at date \( t \).

Financial markets are such that there is short-term riskless borrowing. The short-rate for date \( t \) is \( r_t \), a non-negative \( \mathcal{F}_t \)-measurable random variable, such that one dollar
invested at $t$ returns $1 + r_t$ risklessly at date $t + 1$.\(^8\) For each pair of dates $s, t$ with $s < t$, let $R_{s,t}$ denote the gross return on short-term riskless borrowing at date $s$ for the period to date $t$. In general, $R_{s,t} = (1 + r_s)(1 + r_{s+1}) \cdots (1 + r_{t-1})$. $R_{s,t}$ results from borrowing risklessly one dollar at date $s$ and rolling it over in short-term riskless borrowing repeatedly until date $t$. Since $r_t$ is $\mathbb{F}_t$-measurable, it follows that $R_{s,t}$ is $\mathbb{F}_{t-1}$-measurable, but not before. For any $t$, define $R_{s,t} \equiv 1$.

For ease of interpretations it will be convenient to add an artificial date $T + 1$ to the problem with $Y_{T+1} \equiv D_{T+1} \equiv r_T \equiv 0$. The reasons why will become clear in Section 4.

The manager of the firm must decide when to undertake the venture, i.e., she must choose a time $\tau : \Omega \to \{0, 1, \ldots, T, T+1\}$ such that $\{\tau = s\} \equiv \{\omega : \tau(\omega) = s\} \in \mathbb{F}_s$, $s = 0, 1, \ldots, T$. Such a strategy is a stopping time. For such a stopping time $\tau$, the value of the (unlevered) venture net of invested capital is $X_t$ where $X_{\tau}(\omega) = X_s(\omega)$ when $\tau(\omega) = s$. The event $\{\tau = t\} \cap \{X_t > 0\}$ can be interpreted as the event of exercise or undertaking the venture. The event $\{\tau = t\} \cap \{X_t = 0\}$ can be interpreted as the event of not undertaking the venture. We make these interpretations precise when we characterize the optimal strategies in the next section.

As stated above, we assume the economic setting as described in Myers (1977, Section 4.1) with regard to the existence of valuable real options, what we call here timing ventures. It is assumed that undertaking or not undertaking the venture will not change prices in financial markets and the values $Y_t, K_t, D_t, r_t$ are predicated on this assumption. In this sense, we are talking about small timing ventures.\(^9\) In the absence of
arbitrage there exists a risk-neutral measure $P^*$, equivalent to $P$, such that for any
security paying a dividend stream $\{\delta_j\}$, the price of this security at date $t$, ex the date $t$
dividend, is

$$S_t = E^*_t \left( \sum_{j=t+1}^T \frac{\delta_j}{R_{t,j}} \right),$$  \hspace{1cm} (1)

where $E^*$ denotes expectation under $P^*$, and likewise $E^*_t(x) = E^*(x|\mathbb{F}_t)$, the conditional
expectation under $P^*$ given $\mathbb{F}_t$. Alternatively, we could assume that all the investors,
including shareholders, senior claimants, and managers in this world are risk neutral. In
this case $P^* = P$ and (1) still holds. The absolute values of all random variables in this
paper have finite expectation with respect to $P^*$.

3 Optimal Strategies

For each date $t$ let $C_t$ denote the set of stopping times $\tau$ such that $\tau \geq t$, a.s., the
set of strategies that delay undertaking the venture until at least date $t$. We denote $C_0$
by just $C$. A value maximizing manager will choose $\tau \in C$ so as to maximize the value
of the venture $V_t = E^* \left( \frac{X_\tau}{R_{0,\tau}} \right)$, where as the residual claimants, owners, in a levered
venture would prefer the manager choose $\tau \in C$ to maximize $\tilde{V}_t = E^* \left( \frac{\tilde{X}_\tau}{R_{0,\tau}} \right)$.

Each of these problems is an optimal stopping problem. Solutions to these
problems are not unique in general, but there are canonical solutions that are easy to
describe by dynamic programming arguments. We will describe the canonical solution to
the value maximizing problem. Similar descriptions obtain for the stopping problem for
levered equity considered here and for other problems considered in the rest of the paper. We will try to compare these canonical solutions for levered and unlevered equity in the next section.

Dynamic programming works backward. Define successively \( \gamma_{T+1}, \gamma_T, \gamma_{T-1}, \ldots, \gamma_1 \), by setting

\[
\gamma_{T+1} = X_{T+1}, \tag{2}
\]

\[
\gamma_t = \max \left[ X_t, E_t^* \left( \frac{\gamma_{t+1}}{R_{t,t+1}} \right) \right], \quad t = T, \ldots, 1, 0. \tag{3}
\]

For each \( t \), let \( \tau_t \) be the first date \( s \geq t \) such that \( X_s = \gamma_s \). By (2) such a date \( s \) exists.

Note that this does not depend on there being an artificial date \( T+1 \), because by (2), (3) for date \( T \) has \( \gamma_T = X_T \), which would be (2) with no artificial date \( T+1 \).

**Theorem 1.** For every \( t, \tau_t \in C_t \), and

\[
E^*_t \left( \frac{X_{\tau_t}}{R_{\tau_t,t}} \right) = \gamma_t \geq E^*_t \left( \frac{X_s}{R_{t,s}} \right), \quad \forall s \in C_t. \tag{4}
\]

Since \( R_{0,t} \) is \( \mathbb{F}_{t-1} \)-measurable and \( R_{0,s} = R_{0,t} R_{t,s} \) for \( s \in C_t \), we have from (4) that for each \( t \),

\[
E^*_t \left( \frac{X_{\tau_t}}{R_{0,\tau_t}} \right) \geq E^*_t \left( \frac{X_s}{R_{0,s}} \right), \quad \forall s \in C_t,
\]

which implies the same thing for the unconditional expectations

\[
E^* \left( \frac{X_{\tau_t}}{R_{0,\tau_t}} \right) \geq E^* \left( \frac{X_s}{R_{0,s}} \right), \quad \forall s \in C_t. \tag{5}
\]
The inequality in (5) says that the market value, as of today, of the venture following the strategy $\tau_t$ is greater than the market value of the venture following any other strategy that postpones the undertaking to at least date $t$, i.e., $\tau_t$ maximizes the market value of the venture among those exercise strategies in $C_t$. It follows that $\tau_0$ is the strategy that maximizes the market value of the venture. We call this strategy $\tau^*$. 

The sequence $\gamma_t$, defined in (2) and (3) is referred to commonly as the Snell envelope, for Snell’s (1952) original work on optimal stopping. We interpret $\gamma_t$ as the time value of the venture at date $t$. The time $\tau^*$ is the first date $t$ such that the intrinsic value of the venture $X_t$ equals its time value $\gamma_t$. This is the canonical solution to the problem of maximizing the market value of the venture.

The Snell envelope is useful in characterizing the optimal timing strategy. It is useful as well in determining when the venture finishes out of the money, which we do below. The following result records the optimality of $\tau^*$ and a necessary condition of this optimality. Recall the notation at the beginning of this section that the market value of the strategy $\tau \in C$ is $V_\tau = E^*_\tau \left( \frac{X_\tau}{R_{0,\tau}} \right)$.

**Theorem 2.** $V_{\tau^*} = \max_{\sigma \in C} V_{\sigma}$ and

$$E^*_\tau \left( \frac{X_\tau}{R_{\tau,\tau}} \right) \leq X_{\tau^*} \text{ on } \{\tau^* = t \leq \tau\},$$

for every $\tau \in C$. If $\tau^* \in C$ and $V_{\tau^*} = \max_{\sigma \in C} V_{\sigma}$, then $\tau^* \leq \tau$.

Condition (6) is the intuitive result that says when the market value of the venture is at the optimum, the market value of any strategy that continues past this time is less.
The inequality is weak, however, and hence $\tau^*$ may not be the only optimal strategy. The second statement of Theorem 2 assures us that $\tau^*$ is the minimal optimal strategy.

It is tempting to refer to the time $\tau^*$ as the earliest time to optimally exercise the venture. But it may be the earliest time to decide to never undertake the venture. This is the time the venture finishes out of the money. Define $A_t^* = \{\tau^* = t\} \cap \{X_t = 0\}$. We want to interpret this event as the event “the venture finishes out of the money at date $t$.” The following result provides motivation for this interpretation.

**Theorem 3.** On the event $A_t^*$, $X_{t+k} = \gamma_{t+k} = 0$, $k = 0,1,\ldots,T+1-t$.

Based on Theorem 3, on $A_t^*$ the value of the venture is zero in the most meaningful sense of the term “value of the venture.” It may be that the intrinsic value of the venture at date $t$, $X_t$, is zero in the sense that the option at date $t$ is simply out of the money. But of course the option may still have time value, i.e., it may be that $\gamma_t > 0$.

The event $A_t^*$ is the event where at date $t$ both the intrinsic value $X_t$ of the venture and the time value $\gamma_t$ of the venture are zero. The intuition behind Theorem 3 is that when a non-negative supermartingale hits zero it stays at zero forever. By the recursion (2), (3), the time value $\gamma_t$ is the smallest supermartingale that is larger than $X_t$.

We let $A^* = \bigcup_{t=0}^T A_t^*$. Then $A^*$ is the event in which the venture finishes out of the money, the event in which the venture is foregone. It follows that $A^*$, the complement of this event, is the event in which it is optimal to undertake the venture. Indeed, $A^{*c} = \bigcup_{t=0}^T \{\tau^* = t\} \cap \{X_t > 0\}$.
Theorems 1 and 2 and 3 apply as well to the problem with \( X_i \) replaced by the value of the residual claim \( \tilde{X}_i \). This results in variables \( \tilde{\tau}, \tilde{\tau}, \tilde{\gamma}, \), where \( \tilde{\tau} = \tau_0 \) is the canonical solution to the problem of maximizing the market value of the residual (owners’) claim in the venture. The events \( \tilde{A}_t = \{ \tilde{\tau} = t \} \cap \{ \tilde{X}_t = 0 \} \), \( \tilde{A} = \bigcup_{t=0}^T \tilde{A}_t \) and \( \tilde{A}^c \) have the same interpretations as above but for the levered venture. In the next section we will compare the canonical solutions to the levered and the unlevered ventures.

4 Comparing Optimal Strategies

Intuition, based on Myers’ (1977) insights and results such as those in Mauer and Ott (2000), suggests that \( \tau^* \leq \tau \). But this is not the case, because these timing strategies have two components, the optimal time to exercise and the time the venture finishes out of the money. As we stated earlier, this is just a matter of interpretation. The statement is true of appropriate modifications of these timing strategies. We do this below in Corollary 5.

First, we need the following regularity assumption on the behavior of the value of the prior claim to venture value to rule out possibilities that there is no underinvestment.

**Hypothesis D.** The prior claim to venture value is

(i) significant in that \( D_t \geq 0, t = 0,1,\ldots,T \); and

(ii) declining in time in that \( D_t \geq E_t \left( \frac{D_{t+1}}{R_{t+1}} \right) \), \( t = 0,1,\ldots,T - 1 \).

The prior claim to value being significant is the condition that \( D_t \geq 0 \) for every date \( t \). Recalling that \( D_t \) includes the change in market value of the prior claim, Hypothesis D (i) rules out pure “risk shifts” where the only effect of undertaking the
venture is a decrease in the value of a creditor’s claim because of an increase in risk of the assets backing that claim. It is clear that in such instances there may be no underinvestment, and may even be overinvestment. See Myers (1977, Section 4.2.3) for a discussion of this case.

There also may be no underinvestment if there is value leakage sufficient to induce early exercise of the levered venture. Increase in the exercise price of an American call option is a form of value leakage. If the exercise price increases over the life of the option enough, early exercise may be optimal. Relative to an unlevered venture the value of $D_t$ is an increase in the exercise price over the life of the venture. It turns out that if the exercise price increases at a rate that is no larger than the riskless rate then early exercise will not be optimal. The inequality in Hypothesis D (ii) is equivalent to the assumption that $D_t R_{t,t+1} \geq E_t^s \left( D_{t+1} \right)$, the needed restriction on the rate of increase of the exercise price.

In these respects, we feel that Hypothesis D is a weak regularity requirement, but one sufficient to frame the underinvestment problem. It should be noted that this hypothesis is satisfied by the simple but important case where the $D_t$ arise from a fixed positive promised payment to a creditor and the discounted value process $Y_t$ is a martingale under $P^*$. A numerical example with this feature is given below.

The following theorem gives the comparisons of the two components of the optimal strategies of equity in an unlevered venture and an equivalent but levered venture. Here and elsewhere $I_A$ is the indicator of the event $A$. 

Theorem 4. Under Hypothesis D, \( A^* \subset \tilde{A} \), \( I_{A^*} \tau \leq I_{\tilde{A}} \tau^\ast \), and \( I_{A^*} \tau^\ast \leq I_{\tilde{A}} \tilde{\tau} \). As a consequence, 
\[
E \left( \frac{X_{\tau}}{R_{0,\tau}} \right) \leq V_{\tau} \leq V_{\tau^*}.
\]

The first part of Theorem 4 states that if the unlevered venture finishes out of the money, then so does the levered venture. The second statement says that in the event that the unlevered venture finishes out of the money, the levered venture finished out of the money earlier (no later). The first part also can be read as, if levered equity undertakes the venture, then so does unlevered equity. The third statement of the theorem says that in the event that levered equity undertakes the venture, unlevered equity undertakes it sooner.

The last statement of Theorem 4 says that the value of the venture following the optimal strategy of levered equity is less than the value of the venture following the stopping time \( \tilde{\tau} \) which of course is less than the value of the venture following the value maximizing strategy. In this computation, we had to multiply the payoff \( X_{\tau} / R_{0,\tau} \) by the indicator of the event in which levered equity actually undertakes the venture \( \tilde{A}^\ast \), because the time \( \tilde{\tau} \) does indicate that the levered venture finishes out of the money on \( \tilde{A} \) and there \( \tilde{X}_{\tau} = 0 \), but the unlevered venture finishes out of the money only on the subset \( A^* \) of this event. On the event \( \tilde{A} \cap A^\ast \), the levered venture finishes out of the money, but the unlevered does not, i.e., \( X_{\tau} > 0 \). As a consequence, 
\[
E \left( \frac{X_{\tau}}{R_{0,\tau}} \right) \leq V_{\tau} \leq V_{\tau^*}.
\]

In valuing the venture simply following \( \tau \) would lead to more value on the event \( \tilde{A} \), when levered equity finishes out of the money.
The event $\bar{A} \cap A^c$ is one interpretation of the event that Myers refers to in the following: “The firm financed with risky debt will, in some states of nature, pass up valuable investment opportunities . . .” Of course on $\bar{A}^c$, levered equity undertakes a valuable investment opportunity, but does so later than unlevered equity, and in this sense passes up, in time, the valuable investment opportunity.

Both interpretations of Myers’ statement are correct. When a venture finishes out of the money can be interpreted as the investor or firm never exercising the option to invest. Indeed, we can redefine the optimal strategy to be beyond the horizon $T$ in that event. For example, let

$$v^* = I_A (T+1) + I_{A^c} \tau^*.$$  \hspace{1cm} (7)

Then $v^*$ is a stopping time and it equals the optimal exercise time whenever it is optimal to exercise; otherwise it waits till the expiration of the option and allows it to expire unexercised. Similarly we can let

$$\bar{v} = I_A (T+1) + I_{A^c} \bar{\tau}.$$  \hspace{1cm} (8)

This stopping time is the optimal exercise time for the levered venture when it is optimal to exercise this venture; otherwise it waits till the expiration of the option and lets it expire unexercised. The following is a corollary of Theorem 4.

**Corollary 5.** $V_\tau = V_\tau^*$, $\bar{V}_\tau = \bar{V}_\tau^*$, and under Hypothesis D, $v^* \leq \bar{v}$.

The first two equalities indicate the multiplicity of solutions to the stopping problems characterized in Theorems 1 and 2. It is in the sense of the last part of Corollary 5 that the incentives of equity in a levered timing venture is to wait past the value maximizing time to undertake the venture, sometimes forever.
We present a simple numerical example to illustrate the results here. There are two periods and three dates. Consider a firm whose only asset is a timing venture whose values are described in the following value tree.

The values in the tree are the gross values if the venture is undertaken at those points in time. For example, if the venture is undertaken at date $t = 0$, the venture is worth $Y_0 = 100$ gross of required investment. If it is undertaken at date $t = 1$ following an initial “up” move, the venture is worth $Y_1 = 180$, etc. Here we can take $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. In the now familiar language of binomial trees, we denote by $\omega_1$ the path in the tree of “up” followed by “up”, $\omega_2$ is the path in the tree of “up” followed by “down”, $\omega_3$ is the path “down” followed by “up”, and $\omega_4$ is “down” followed by “down”. The following table summarizes the gross values.

<table>
<thead>
<tr>
<th>Path</th>
<th>Values of</th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Y_0$</td>
<td>$Y_1$</td>
<td>$Y_2$</td>
<td></td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>100</td>
<td>180</td>
<td>324</td>
<td></td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>100</td>
<td>180</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>100</td>
<td>60</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>100</td>
<td>60</td>
<td>36</td>
<td></td>
</tr>
</tbody>
</table>
These values are consistent with a risk-neutral probability of .4 of an “up” move anywhere in the tree. So $P^* \{ \omega_1 \} = .4(.4) = .16$, $P^* \{ \omega_2 \} = .4(.6) = .24$, $P^* \{ \omega_3 \} = .6(.4)$ = .24, and $P^* \{ \omega_4 \} = .6(.6) = .36$. We assume that interest rates are deterministic and equal to 8% per period. So $R_{0,t} = (1.08)^t$, $t = 0, 1, 2$. Then the discounted value process $Y_t$ is a martingale under $P^*$ in that $Y_t = E_t^* \left( \frac{Y_{t+1}}{R_{t+1}} \right)$, $t = 0, 1$.

Suppose that to undertake the venture requires an initial investment. Of course if the investment is fixed, the venture is an American call option on an asset that has no value leakage and hence the optimal exercise strategy is to wait till date $t = 2$ and exercise the option when it is in the money. By Theorem 4, the optimal exercise strategy for a levered venture is also to wait till date $t = 2$ and exercise when the residual claim is in the money.

Of course the above reasoning only applies to the exercise component of the optimal timing strategy. It may well be that even for American call options it finishes out of the money before the terminal date. To illustrate all aspects of Theorems 1-4 and Corollary 5, we induce earlier exercise by assuming that the required investment $K_t$ for undertaking the venture at date $t$ is as follows. To undertake the venture at date $t = 0$ or at $t = 1$ the required investment is 100, regardless of the state. To undertake at $t = 2$ the required investment is 100 in states $\omega_2, \omega_3,$ and $\omega_4$, but it jumps up to 145 in state $\omega_1$. So $K_0 \equiv K_1 \equiv 100$ and $K_2(\omega_i) = 100, i = 2, 3, 4$, but $K_2(\omega_1) = 145$.

Then the values of interest here are the net present values $X_t = (Y_t - K_t)^+, t = 0, 1, 2$. These values are summarized in the following table.
<table>
<thead>
<tr>
<th>Path</th>
<th>$X_0$</th>
<th>$X_1$</th>
<th>$X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>0</td>
<td>80</td>
<td>179</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>0</td>
<td>80</td>
<td>8</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Given these values, it is straightforward to determine the value maximizing time $\tau^*$. Since $X_0 = 0$, $\tau^* > 0$ since there is value in waiting. Following an initial “down” move, it is clearly in the interest of unlevered equity to wait till $t = 2$ since $X_1 = 0$ and waiting has the value $\gamma_1 = E^* \left( \frac{\gamma_2}{R_{t,2}} \right) = \frac{.4(8) + .6(0)}{1.08} = 2.96$. This is an event when the venture is out of the money but has time value. The venture is then undertaken in a subsequent “up” move but it finishes out of the money in a subsequent “down” move. So $\tau^*(\omega_3) = \tau^*(\omega_1) = 2$.

To determine the value of $\tau^*$ in the other instances, note that following an initial “up” move, the value of waiting one more period is $E^* \left( \frac{\gamma_2}{R_{t,2}} \right) = \frac{.4(179) + .6(8)}{1.08} = 70.74 < 80 = X_1$. Thus following an initial “up” move, $\gamma_1 = X_1$ and this is the first time that happens, so $\tau^*(\omega_1) = \tau^*(\omega_2) = 1$. The venture is undertaken since $X_1 > 0$.

It follows that

$$V_{t^*} = E^* \left( \frac{X_{t^*}}{R_{0,t^*}} \right) = E^* \left( I_{[t^*=1]} \frac{X_1}{R_{0,1}} + I_{[t^*=2]} \frac{X_2}{R_{0,2}} \right) = \frac{.4(80)}{(1.08)} + \frac{.24(8) + .36(0)}{(1.08)^2} = 31.28.$$  

Notice that the venture is undertaken in every state it has positive NPV, the event $A^* = \{\omega_1, \omega_2, \omega_3\}$. The venture finishes out of the money in $A^* = \{\omega_4\}$. 

17
Suppose the venture is levered with a debt claim that has a promised payment of 50, so \( D_t \equiv \min \{50, Y_t \}, t = 0,1,2 \). We are in the case described in Lemma A.2 in the Appendix and hence the \( D_t \) satisfy Hypothesis D. The values of levered equity \( \tilde{X}_t = (Y_t - K_t - D_t)^+ \) are given in the following table.

<table>
<thead>
<tr>
<th>Path</th>
<th>( X_0 )</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_1 )</td>
<td>0</td>
<td>30</td>
<td>129</td>
</tr>
<tr>
<td>( \omega_2 )</td>
<td>0</td>
<td>30</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \omega_4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Obviously \( \tau > 0 \), there is some value to waiting. However, following an initial “down” move, the value of waiting is zero and so is \( \tilde{X}_1 \). Indeed, \( \tilde{X}_2 = \gamma_2 = 0 \) in this event, and hence \( \tilde{X}_1 = \gamma_1 = 0 \), and this is the first time this happens, i.e., \( \tau (\omega_3) = \tau (\omega_4) = 1 \).

Following an initial “up” move, the value of waiting is
\[
E_1 \left( \frac{\gamma_2}{R_{1,2}} \right) = \frac{.4(129) + .6(0)}{1.08} = 47.78 > 30 = \tilde{X}_1 . \quad \text{As a consequence} \quad \gamma_1 = E_1 \left( \frac{\gamma_2}{R_{1,2}} \right) > \tilde{X}_1 ,
\]
and \( \tau (\omega_1) = \tau (\omega_2) = 2 \). Levered equity will wait till date \( t = 2 \) to undertake this venture. Following a second up move, the venture will be undertaken, but it finishes out of the money in a subsequent “down” move.

The event in which the levered venture is not undertaken is the event in which this venture finishes out of the money, the event \( \tilde{A} = \{\omega_2, \omega_3, \omega_4\} \supset A' = \{\omega_4\} \). Note that \( \tau (\omega_3) = 1 < 2 = \tau' (\omega_3) \), so levered equity “passes up a valuable investment opportunity” before the value maximizing time. The event in which the levered venture is undertaken
is $\widetilde{A}^c = \{\omega_i\} \subset A^c$ as indicated in Theorem 4. The event $\widetilde{A} \cap A^c = \{\omega_2, \omega_3\}$ is the event in which levered equity foregoes a positive NPV venture.

Indeed, there is value loss here for

$$E^*(I_{\overline{A}} \frac{X_t}{R_{0,t}}) = E^*(I_{\overline{A}} \left( I_{[\tau = 1]} \frac{X_1}{R_{0,1}} + I_{[\tau = 2]} \frac{X_2}{R_{0,2}} \right)) = \frac{.16(179)}{(1.08)^2} = 24.55 < V_\tau.$$

Here

$$V_\tau = E^*(\frac{X_t}{R_{0,t}}) = E^*(I_{[\tau = 1]} \frac{X_1}{R_{0,1}} + I_{[\tau = 2]} \frac{X_2}{R_{0,2}}) = \frac{.16(179) + .24(8)}{(1.08)^2} = 26.20,$$

overstating the value of following the optimal strategy of levered equity.

The modified strategies of (7) and (8) are compared in the following table.

<table>
<thead>
<tr>
<th>Path</th>
<th>Values of $\nu^+$ of $\hat{\nu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>$\tau^*(\omega_1) = 1$</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>$\tau^*(\omega_2) = 1$</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>$\tau^*(\omega_3) = 2$</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>3</td>
</tr>
</tbody>
</table>

5 Increase in Leverage

The incentives of owners of a levered timing venture to wait beyond the value maximizing time to undertake the venture result from them being risk loving relative to owners of an unlevered venture. As leverage increases owners of levered ventures get more risk loving.

To make things precise, let $D^1_t, D^2_t$ denote the prior claim to venture value arising from two different levels of leverage. Let $\hat{X_t} = (Y_t - K_t - D^i_t)^+$, the value of the residual
claim in the levered venture with leverage \( D_i^j, i = 1, 2 \). For the sake of economy in describing these we may just refer to them as venture \( i = 1, 2 \).

Similarly, the relevant variables of Theorems 1 and 2 are denoted \( \tau^i, \gamma^i, \eta^i \), where \( \tau^i = \tau_0^i \) is the canonical solution to the problem of maximizing the market value of the residual (owners’) claim in venture \( i = 1, 2 \). The events \( \tilde{A}_i = \{ \tau^i = t \} \cap \{ \tilde{X}_i^j = 0 \} \) and \( \tilde{A}^i = \bigcup_{t=0}^T \tilde{A}_i \) and \( \tilde{A}^i \) have the same interpretations as above but for the levered venture \( i = 1, 2 \).

**Theorem 6.** Assume that \( D_i^2 - D_i^1, t = 0, 1, \ldots, T \), satisfy Hypothesis D. Then \( \tilde{A}_i \subset \tilde{A}_2 \),

\[
I_{\tilde{A}_i^2} \leq I_{\tilde{A}_i^1}, I_{\tilde{A}_i^2} \leq I_{\tilde{A}_i^1}, \text{ and } E^x \left( I_{\tilde{A}_i^1} \frac{X_{\tau^1}}{R_{0,\tau^1}} \right) \leq E^x \left( I_{\tilde{A}_i^1} \frac{X_{\tau^1}}{R_{0,\tau^1}} \right) \leq V_{\tau^1}.
\]

The hypothesis of Theorem 6 that the difference \( D_i^2 - D_i^1 \) satisfies Hypothesis D includes a sense in which the leverage in venture 2 is greater than the leverage in venture 1. This is Hypothesis D(i).

Theorem 6 is really a general restatement of Theorem 4, for the case of \( D_i^1 \equiv 0, t = 0, 1, \ldots, T \) is Theorem 4. The first part of Theorem 6 states that if the venture with lower leverage finishes out of the money, then so does venture with the higher leverage. The second statement says that in the event that venture with low leverage finishes out of the money, the venture with higher leverage finished out of the money earlier (no later). The first part also can be read as, if the venture with high leverage undertakes the venture, then so does the venture with lower leverage. The third statement of theorem says that in the event that the venture with high leverage undertakes the
venture, the venture with low leverage undertakes it sooner. The last part of Theorem 6 states that there is loss (no gain) of value using debt and this loss (weakly) increases with leverage.

Now let $V^1$ and $V^2$ be defined as in (8) of the previous section. Then as in the case of Corollary 5, we have that

**Corollary 7.** $\tilde{V}_t^1 = \tilde{V}_t^2$, $\tilde{V}_t^2 = \tilde{V}_t^2$, and if $D^t_i - D^t_i, t = 0, 1, \ldots, T$, satisfy Hypothesis D, then $V^1 \leq V^2$.

Consider again the numerical example of Section 4. In that example, the firm’s only asset is a timing venture and the prior claim to value was a promised payment of 50. We had $\tilde{\tau} (\omega_1) = \tilde{\tau} (\omega_3) = 1$ and $\tilde{\tau} (\omega_2) = \tilde{\tau} (\omega_4) = 2$ and there was loss of value with that much leverage. But if leverage is sufficiently low, we can get something better. As a function of the promised payment $D$, after an initial “up” move the value of waiting till $t = 2$ is $\frac{.4(179 - D)}{1.08}$ and the payoff to undertaking the venture at date $t = 1$ is $80 - D$.

Setting these equal and solving for $D$ gives a breakeven value of 21.76. So take $D^1 = 20$ and $D^1_i = \min\left[D^1, Y^i_t\right], t = 0, 1, 2$. Then $\tilde{\tau}^1 (\omega_1) = \tilde{\tau}^1 (\omega_2) = 1$ and $\tilde{\tau}^1 (\omega_3) = \tilde{\tau}^1 (\omega_4) = 1$, but $\tilde{\tau}^1 (\omega_3, \omega_4) = 1$, but $\tilde{\tau}^1 (\omega_3, \omega_4) = 1$. Also $\tilde{\tau}^1 (\omega_3) = \tilde{\tau}^1 (\omega_4) = 1$, but $\tilde{\tau}^1 (\omega_3, \omega_4) = 1$. Evaluating the venture value with debt $D^i$,

$$E^T \left( I_{\tilde{\tau}^i} X_{\tilde{\tau}^i} \right) = E^T \left( I_{\tilde{\tau}^i} X_{\tilde{\tau}^i} \right) \frac{.4(80)}{1.08} = 29.63.$$  

Letting $D^2 = 50$ and $D^2_i = \min\left[D^2, Y^i_t\right], t = 0, 1, 2$, $\tilde{\tau}^2 (\omega_1) = \tilde{\tau}^2 (\omega_2) = 1$ and $\tilde{\tau}^2 (\omega_3) = \tilde{\tau}^2 (\omega_4) = 2$, and we have that
Check that $D^2_i - D^1_i$ satisfies Hypothesis D. The values are presented in the following table.

<table>
<thead>
<tr>
<th>Path</th>
<th>$D^2_0 - D^1_0$</th>
<th>$D^2_1 - D^1_1$</th>
<th>$D^2_2 - D^1_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>30</td>
<td>30</td>
<td>16</td>
</tr>
</tbody>
</table>

Notice that incentives to not underinvest are improved with less leverage. As the example illustrates, however, these incentives are not restored to full efficiency until the level of the promised payment gets below 8. Indeed $\tilde{A}^l = \{\omega_3, \omega_4\}$, where the venture finishes out of the money, for all levels of $D^1$, $X_2(\omega_3) = 8 \leq D^1 < 21.76$. So here the riskiness of the debt has to be judged relative to the lowest net venture value in which the venture would be undertaken by the unlevered firm.

### 6 Conclusions

Owners of levered timing ventures have incentives to postpone implementation of those ventures beyond the value maximizing time. At the extreme they may forego such ventures as indicated in Myers (1977). Viewing growth opportunities as timing ventures, owners of firms with such opportunities or managers of firms with such opportunities acting in shareholders’ interest will not exercise these options optimally in terms of value maximization.
The reason for these perverted incentives is the convex form of the residual claim. Owners of levered ventures are risk loving relative to owners of equivalent but unlevered ventures. Their incentives to wait longer may result in loss of value. These incentives to wait get worse the greater the leverage.

Aligning managers interests with shareholders by giving them call options on the stock of their firm can have the same effect if these managers make the timing decisions for undertaking growth opportunities. We would expect that firms with substantial growth opportunities would use little risky debt to avoid the non-optimal exercise of these options. Indeed all the cautions about potential solutions to the underinvestment problem and their costs examined in detail in Myers (1977) still apply here.

The implication for optimal capital structure for timing ventures is clear. Levered timing ventures are less valuable than unlevered ones, so the optimal capital structure is no leverage.

Appendix

Proofs and references for proofs of results in the text are presented in this appendix. All properties of conditional expectations used here and in the text can be found in Williams (1991, Section 9.7).

Proof of Theorem 1. Fix $t, 1 < t \leq T$, take any $s \in C_{t-1}$ and let $s' = \max[s, t]$. Then $s' \in C_t$ and since $R_{t-1,s'} = 1$,

$$\frac{X_s}{R_{t-1,s}} = X_{t-1}I_{[s=t]} + \frac{X_{s'}}{R_{t-1,s'}} I_{[s' \geq t]},$$

(A.1)

where $I_A$ is the indicator of the event $A$. With (A.1) the induction argument in Chow, Robbins, and Siegmund (1991, Theorem 3.2) goes right through, to establish (4).[]
Proof of Theorem 2. The first part follows from (5). The second and third parts follow from Chow, Robbins, and Siegmund (1991, Lemma 3.2), since here \( \sup_{\sigma \in C} V_\sigma < \infty \).

Proof of Theorem 3. By definition of \( \tau^* \), \( \gamma_t = 0 \) on \( A^*_t \in \mathbb{F}_t \). It follows by (3) that

\[
I^*_{A^*} \gamma_t \geq I^*_{A^*} \left( \frac{\gamma_{t+1}}{R^*_{t+1}} \right) = E^*_t \left( I^*_{A^*} \frac{\gamma_{t+1}}{R^*_{t+1}} \right). \]

Then \( 0 = \int E^*_t \left( I^*_{A^*} \frac{\gamma_{t+1}}{R^*_{t+1}} \right) dP^* = \int I^*_{A^*} \frac{\gamma_{t+1}}{R^*_{t+1}} dP^* \)

implies that \( I^*_{A^*} \gamma_{t+1} = 0 \). Using (3) again for each later date, it then follows that \( I^*_{A^*} \gamma_{t+k} = 0, k = 0,1,\ldots,T-t \), and hence that \( I^*_{A^*} X_{t+k} = 0, k = 0,1,\ldots,T-t \), from (2) and (3).

We need the following result for the proof of Theorem 4.

**Lemma A.1.** Under Hypothesis D (ii), \( D_t \geq E^*_t \left( \frac{D_j}{R_{t,j}} \right) \), for all \( s \in C_t, t = 1,\ldots,T \).

Proof: Given \( t \) and \( s \in C_t \), we have

\[
E^*_t \left( \frac{D_s}{R_{t,s}} \right) = E^*_t \left( \sum_{j=1}^{T} I_{\{s=j\}} \frac{D_j}{R_{t,j}} \right) = E^*_t \left( \sum_{j=1}^{T-1} I_{\{s=j\}} \frac{D_j}{R_{t,j}} + I_{\{s=T\}} \frac{1}{R_{t,T-1}} E^*_t \left( \frac{D_T}{R_{T-1,T}} \right) \right)
\]

\[
\leq E^*_t \left( \sum_{j=1}^{T-2} I_{\{s=j\}} \frac{D_j}{R_{t,j}} + I_{\{s=T-1\}} \frac{D_{T-1}}{R_{t,T-1}} \right)
\]

\[
E^*_t \left( \sum_{j=1}^{T-2} I_{\{s=j\}} \frac{D_j}{R_{t,j}} + I_{\{s=T-1\}} \frac{1}{R_{t,T-2}} E^*_t \left( \frac{D_{T-2}}{R_{T-2,T-1}} \right) \right)
\]

\[
\leq E^*_t \left( \sum_{j=1}^{T-2} I_{\{s=j\}} \frac{D_j}{R_{t,j}} + I_{\{s=T-2\}} \frac{D_{T-2}}{R_{t,T-2}} \right)
\]

\[
\ldots \leq E^*_t \left( I_{\{s=T\}} \frac{D_T}{R_{t,T}} \right) = D_t.
\]
The equalities follow from the law of iterated conditional expectation and the fact that \( \{s \geq j\} \in \mathbb{F}_{j-1} \). The inequalities follow from Hypothesis D.[]

Applying this result for \( t = 0 \) gives us that \( D_0 \geq E^* \left( \frac{D_s}{R_{0,s}} \right) \), for all \( s \in C \). So if prior claimants were given the right to decide when to undertake the venture using shareholders’ capital and they decided by maximizing the market value of their claim to venture value, their optimal choice \( \tau^* \) would always be \( \tau^* = 0 \). Of course when \( D_0 = 0 \), prior claimants would forego the venture, and obviously do so before the value maximizing time.

The claim about the simple but important case when Hypothesis D holds is the following.

**Lemma A.2.** Suppose that for some \( D > 0 \), \( D_t = \min \{D, Y_t\} \) and suppose that the discounted value process is a martingale in that \( Y_t = E_t^* \left( \frac{Y_{t+1}}{R_{t+1}} \right), t = 0,1,\ldots,T - 1 \). Then \( D_t \) satisfies Hypothesis D.

Proof: Clearly Hypothesis D (i) is satisfied. For \( t \) given

\[
E_t^* \left( \frac{D_{t+1}}{R_{t+1}} \right) = E_t^* \left( \frac{D \wedge Y_{t+1}}{R_{t+1}} \right) = E_t^* \left( \frac{D \wedge Y_{t+1}}{R_{t+1}} \right) \leq \frac{D \wedge E_t^* (Y_{t+1})}{R_{t+1}} \leq \frac{D \wedge E_t^* (Y_{t+1})}{R_{t+1}} = D \wedge Y_t = D_t,
\]

where the first equality follows by definition of \( D_{t+1} \), the second because \( R_{t+1} = 1 + r_t \) is known at date \( t \), the first inequality from Jensen’s inequality for conditional expectations, the next inequality follows from the fact \( 1 + r_t \geq 1 \). Finally, the equality follows from the martingale assumption on the value process \( Y_t \). This establishes Hypothesis D (ii). []
This result is of course the case of a concave transformation of a martingale being a supermartingale. The martingale hypothesis of Lemma A.2 would per force hold by (1) if the value process was the value of a traded asset, but need not hold if the underlying is not a traded asset. Convex transformations of martingales of course are submartingales. It is this property that gives rise to the distorted incentives of levered equity. After we show the techniques for the proofs of Theorems 4 and 6, we treat an interesting case of the prior claim being a proportional share claim. Under the martingale hypothesis, the results of Theorem 4 obtain as well.

Proof of Theorem 4. Under Hypothesis D (i), \( X_t = (Y_t - K_t - D_t)^+ = (X_t - D_t)^+ \leq X_t \). By Theorem 3, \( I_{t^*}X_{r+k} = 0, \ k = 0,1,\ldots,T+1-t \) and hence \( I_{t^*}X_{r+k} = 0, \ k = 0,1,\ldots,T+1-t \). It follows from (2) and (3) that \( I_{t^*}Y_{r+k} = 0, \ k = 0,1,\ldots,T+1-t \). Hence \( I_{t^*}\tau \leq t = I_{t^*}t^* \), establishing the second part of the theorem.

Let \( \bar{B}_t = \{\bar{t} = t\} \cap \{\bar{X}_t > 0\} \) and note that \( \bar{A} = \bigcup_{t=0}^{T} \bar{B}_t \). Then

\[
I_{\bar{t}}\bar{X}_t = I_{\bar{t}}(X_t - D_t) = I_{\bar{t}}\bar{Y}_t \geq I_{\bar{t}}E_t \left( \frac{\bar{X}_t}{R_{t,s}} \right), \forall s \in C_t, \tag{A.2}
\]

where the first equality comes from the fact \( \bar{X}_t > 0 \), the second from the definition of \( \bar{t} \), and the inequality from (4) for the problem with \( X_t \) replaced by \( \bar{X}_t \). Taking \( D_t \) to the other side of this inequality, we get that

\[
I_{\bar{t}}X_t \geq I_{\bar{t}}E_t \left( \frac{X_t}{R_{t,s}} \right) + I_{\bar{t}}D_t \geq I_{\bar{t}} \left( E_t \left( \frac{X_t - D_t}{R_{t,s}} \right) \right)^+ + D_t \\
\geq I_{\bar{t}} \left( E_t \left( \frac{X_t - D_t}{R_{t,s}} \right) + D_t \right)
\]
\[
I_B \left( E\left( \frac{X_t}{R_{t,s}} \right) + D_t - E\left( \frac{D_s}{R_{t,s}} \right) \right) \\
\geq I_{B_s} E\left( \frac{X_t}{R_{t,s}} \right) \forall s \in C_t.
\]  

(A.3)

where the first inequality follows from (A.2), the second from Jensen’s inequality for conditional expectations, the third inequality follows from the positive part, and the last inequality follows from Hypothesis D (ii) and Lemma A.1. It then follows from (A.3) that \( I_{B_t} X_t = I_{B_t} \gamma_t \), by (4) because \( \tau_t \in C_t \). Thus \( I_{B_t} \tau^* \leq t = I_{B_t} \tau^* \), proving the third part of the theorem.

The first part of the theorem follows from the inequalities in the second and third parts. The first inequality in the last part of the theorem follows from the fact that \( I_A \frac{X_t}{R_{0,t}} \geq 0 \), and the second inequality follows from Theorem 2 and the fact that \( \tau \in C \).

Proof of Corollary 5. The first two equalities follow from the fact that \( X_{t^*} = 0 = X_{T+1} \) on \( A^\circ \) and \( X_{t^*} = 0 = X_{T+1} \) on \( A^\circ \). From (7),

\[
v^* = I_{A_t^*} (T + 1) + I_{A^*} \tau^*
\\
= I_{A_t^*} (T+1) + I_{A^* \cap A^*} \tau^* + I_{A^*} \tau^*
\\
\leq I_{A_t^*} (T + 1) + I_{A^* \cap A^*} (T+1) + I_{A^*} \tau^* = v^*,
\]

by (8), where the decomposition in the second equality and the inequality follow from Theorem 4.

Proof of Theorem 6. The proof of Theorem 6 follows just as in the proof of Theorem 4.

By Hypothesis D (i), \( D_t^2 \geq D_t \) implies \( \bar{X}_t^2 = \left( \bar{X}_t^1 - (D_t^2 - D_t^1) \right)^* \leq \bar{X}_t^1 \). The inequality
(A.2) is for the higher debt level $D_t^2$ and only $D_t^2 - D_t^1$ is moved to the other side to produce inequality (A.3) for the lower debt level $D_t^1$. Then use Lemma A.1 to conclude that $D_t^2 - D_t^1 - E_t^s \left( \frac{D_t^2 - D_t^1}{R_{v,s}} \right) \geq 0$.\)

Proof of **Corollary 7.** Follows from Theorem 6 as Corollary 5 does from Theorem 4.\]

Consider now the case when $D_t = \alpha Y_t$ for some fraction $\alpha, 0 < \alpha < 1$. This might be the claim of a venture capitalist who cannot be compelled to contribute additional capital to the venture. In this case, the results of Theorem 4 obtain when the value of the underlying satisfies the martingale hypothesis. Again, we note that this would be the case if the underlying were a traded asset, e. g., if there was a secondary market for the asset. Myers (1977, Section 3.3) discusses the underinvestment problem for this case when the prior claim is a debt claim. The case here extends this to a prior claim that is a proportional claim. The notation of Section 4 applies here.

**Theorem A.3.** Assume that $D_t = \alpha Y_t$, $\tilde{X}_t = \left( (1 - \alpha)Y_t - K_t \right)^+$ and that

$$Y_t = E_t^s \left( \frac{Y_{t+1}}{R_{t+1}} \right), t = 0, 1, \ldots, T - 1.$$ Then $A^* \subset \tilde{A}$, $I_{\tilde{A}} \tau^* \leq I_A \tau^*$, and $I_A \tau^* \leq I_{\tilde{A}} \tau^*$. As a consequence, $E^s \left( I_{\tilde{A}} \frac{X_{\tau^*}}{R_{\tilde{A}, \tau}} \right) \leq V_{s} \leq V_{\tau^*}$.

**Proof:** The non-negativity of $Y_t$ and the martingale hypothesis ensures that the $D_t$ satisfy Hypothesis D. In fact, by Lemma A.1, it follows that $D_t - E_t^s \left( \frac{D_t}{R_{y,s}} \right) = 0$, for all $s \in C_t$. \[ \]
We have that $\bar{X}_t = (X_t - \alpha Y_t)^+ \leq X_t$. The proof proceeds just as in the proof of Theorem 4.[4]

References


**Endnotes**

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1 See Dixit and Pyndyck (1994) and Trigeoris (1996) for a review and presentation of this literature. See also Brennan and Trigeorgis (2000) and Schwartz and Trigeorgis (2001) for a sample of recent contributions.


3 Myers (1977, p. 149).
Galai and Masulus (1976) note this convexity turns levered equity into a call option on the assets of the firm. They attribute the asset substitution problem in this light.

The horizon is the option expiration. The fixed finite horizon is not essential but is convenient in describing optimal strategies. The results established here hold if the horizon or option expiration is an arbitrary stopping time. See Nachman (1975) for this general case. More realistic economic circumstances, such as competition, that determine expiration are beyond the scope of this paper.

For simplicity we ignore taxes. To the extent that any part of the prior claim to value is tax deductible so the after-tax claim to value is less, the leverage is less and the underinvestment problem is less. See Theorem 6 in Section 5 for the effect of changes in leverage.


Riskless borrowing here plays the role of impatience in Nachman (1975), but no result here requires it, i.e., every result here is valid even if \( r_t = 0 \) for all \( t \).

The ventures need not be small if financial markets are complete. See Arnold and Shockley (2003).


Unless explicitly stated to the contrary, all relations among random variables in this paper hold almost surely, abbreviated a.s. With this understood, we henceforth drop this qualifier in the text.

See for example Duffie (2001, pp. 33,184). Only in the finite horizon case can this envelope be defined constructively as in (2) and (3). See Chow, Robbins, and Siegmund (1991, Chapter 4) for the non-constructive definitions and Nachman (1975) for the effect of risk aversion and impatience on the optimal stopping times in this case.

The terms “intrinsic value” and “time value” used here are the familiar terms used in describing option value. See for example Bodie, Cane, and Marcus (2001, pp. 541,542).

It is never the case that \( T+1 \) is the first time this happens.

See the comment following Corollary 5 below.
This holds when the discounted value process is a martingale under $P$ (see Lemma A.1 and the comments that follow it in the Appendix). The act of early exercise saves the increase in the exercise price, but if what you save is less than what you can earn on the saved exercise price, it would not pay to exercise early.

Hypothesis D is the hypothesis that the discounted sequence of prior claim values in the venture is a non-negative supermartingale. See Lemma A.1 in the Appendix and the comments that follow it for one implication of this hypothesis.

This is shown in Lemma A.2 of the Appendix.

Myers, op. cit.

This is the generic example of Trigeorgis (1996, Chapter 5).

For any constant exercise price greater than 108 but less than 324 in this example has the venture finishing out of the money at date $t = 1$ following an initial down move.

This can be made starker. Suppose $K_2(\omega_1) = 30$ instead of 100, but all else is the same. Then the payoffs to levered and unlevered equity remain the same except that $X_2(\omega_4) = 6$. The optimal times $\tau^*$ and $\tau$ remain unchanged, but now $A^* = \emptyset$ and the (unlevered) venture is positive net present value in both $\omega_3$ and $\omega_4$. Yet in both these states levered equity passes up the valuable investment before the value maximizing time.