Learning Objectives

- Calculate the PV and FV in multi-period multi-CF time-value-of-money problems:
  - General case
  - Perpetuity
  - Annuity
- Find the rate of return in multi-period (multi-CF) time-value-of-money problems
  - Adjusting the rate of return:
    - The frequency of compounding
    - Inflation
- Loan amortization schedule

Example

You plan to spend the next four summers abroad. The first summer trip, which is exactly one year away, will cost you $22,000, the second, $27,500, the third, $33,000 and the fourth $35,000.

1. How much should you deposit in your account today (pays 6% interest per annum) so that you will have exactly enough to finance all the trips?
2. If you borrow the money to finance those trips (at 6% interest per annum) and plan to repay it in 5 years when you get your trust fund, how much do you expect to pay?

Present Value (PV) of a CF Stream

\[
PV = \frac{CF_1}{(1+r)^1} + \frac{CF_2}{(1+r)^2} + \ldots + \frac{CF_T}{(1+r)^T}
\]

- \(CF_t\) is the cash flow on date \(t\) (end of year \(t\))
- \(r\) is the cost of capital for one period (one year)
- \(t\) is date index, \(t = 1, 2, 3, \ldots, T\)
- \(T\) is the number of periods (number of years)

Future Value (FV) of a CF Stream

\[
FV = PV \times (1+r)^T
\]

Perpetuity

You invest in a project that is expected to pay $1,200 a month, at the end of the month, forever.

The monthly cost of capital is 1%. What is the present value of this CF stream?
Present Value (PV) of a Perpetuity

<table>
<thead>
<tr>
<th>CF</th>
<th>CF</th>
<th>…</th>
<th>CF</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>t</td>
<td>t</td>
</tr>
</tbody>
</table>

Time

$CF = \text{the SAME CF at the end of EVERY period (year)}$

First CF (start date): end of the first period (date 1)

We get the same CF FOREVER ($T = \infty$, infinity)

$r = \text{the cost of capital for one period (one year)}$

$$PV = \frac{CF}{r}$$

Perpetuity examples

1. Suppose the value of a perpetuity is $38,900 and the discount rate is 12% per annum. What must be the annual cash flow from this perpetuity?

2. An asset that generates $890 a year forever is priced at $6,000. What is the required rate of return?

Annuity

You consider investing in real estate. You expect the property to yield (i.e., generate) rent CFs of $18,000 a year for the next twenty years, after which you will be able to sell it for $250,000.

Your required rate of return is 12% per annum. What is the maximum amount you’d pay for this CF stream?

Present Value (PV) of an Annuity

<table>
<thead>
<tr>
<th>CF</th>
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<th>…</th>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>t</td>
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<td>T</td>
</tr>
</tbody>
</table>

Time

$CF = \text{the SAME CF at the end of EVERY period (year)}$

First CF (start date): end of the first period (date 1)

Last CF (end date): end of the last period (date T)

$T = \text{the number of periods (number of years)}$

$r = \text{the cost of capital for one period (one year)}$

$$PV = \frac{CF}{r} \left[ 1 - \frac{1}{(1+r)^T} \right]$$

Annuity, find the FV

You open a savings account and deposit $20,000 today. At the end of each of the next 15 years, you deposit $2,500.

The annual interest rate is 7%. What will be the account balance 15 years from now?

Annuity, find the PMT

You are trying to borrow $200,000 to buy a house on a conventional 30-year mortgage with monthly payments.

The monthly interest rate on this loan is 0.70%. What is the monthly payment on the loan?
Annuity, find the PMT: Challenge

You plan to retire in 30 years. Then you will need $200,000 a year for 10 years (first withdrawal at t=31). Ten years later you expect to go to a retirement home where you will stay for the rest of your life. To enter the retirement home, you will have to make a single payment of $1,000,000. You can start saving for your retirement in an account that pays 9% interest a year. Therefore, starting one year from now (end of the first year; t =1), you will make equal yearly deposits into this account for 30 years. In 30 years (on date t=30), you expect a deposit of $500,000 to your retirement account from your cash value insurance policy. What should be your yearly deposit into the retirement account?

Adjusting the rate of return

- The frequency of compounding:
  - Quoted (stated) rate
  - Effective rate
  - Always use the effective annual rate to discount annual CFs, effective monthly rate to discount monthly CFs etc.

- The case of inflation:
  - Nominal rate
  - Real rate
  - Always use the nominal rate to discount nominal CFs and the real rate to discount real CFs.

Effective to Effective: Example

The annual interest rate is 8%.
What is the 2-year rate of return on $1?

\[ FV = PV \times \left(1 + r_{\text{effective, 2-year}}\right)^2 = PV \times \left[1 + r_{\text{effective, 2-year}}\right]^2 \]

\[ FV = \$1 \times \left[1 + r_{\text{effective, 2-year}}\right]^2 = \$1 \times \left[1 + r_{\text{effective, 2-year}}\right] \]

\[ [1 + 0.08]^2 = \left[1 + r_{\text{effective, 2-year}}\right] = 1.1664 \]

\[ r_{\text{effective, 2-year}} = 1.1664 - 1 = 0.1664 = 16.64\% \]

Effective to Effective: Formula

\[ r_{\text{effective, n-period}} = n\text{-period effective rate} \]

The return on $1 invested for n periods

\[ r_{\text{effective, n-period}} = \left[1 + r_{\text{effective, 1-period}}\right]^n \]

Effective to Effective: n>1

The effective monthly rate is 1%, what is the effective annual rate?

Since the effective monthly rate is known and there are n=12 months in one year

\[ r_{\text{effective, monthly}} = 1\% = 0.01 \]

\[ 1 + r_{\text{effective, 12-months}} = \left[1 + r_{\text{effective, 1-month}}\right]^{12} \]

\[ 1 + r_{\text{effective, monthly}} = \left[1 + 0.01\right]^{12} \]

\[ r_{\text{effective, annual}} = \left[1 + 0.01\right]^{12} - 1 \approx 0.1268 = 12.68\% \]

Effective to Effective: n>1

The effective monthly rate is 1%, what is the effective quarterly rate?

Since the effective monthly rate is known and there are n=3 months in one quarter

\[ r_{\text{effective, monthly}} = 1\% = 0.01 \]

\[ 1 + r_{\text{effective, 3-months}} = \left[1 + r_{\text{effective, 1-month}}\right]^3 \]

\[ 1 + r_{\text{effective, monthly}} = \left[1 + 0.01\right]^3 \]

\[ r_{\text{effective, quarterly}} = \left[1 + 0.01\right]^3 - 1 = 0.0303 = 3.03\% \]
Effective to Effective: n<1

The effective monthly rate is 1%, what is the effective weekly rate?
Since the effective monthly rate is known and there are 4 weeks in one month or \((1/4)=0.25\) months in one week
\[
r_{\text{effective,monthly}} = 1\% = 0.01
\]
\[
1 + r_{\text{effective,monthly}} = \left[1 + r_{\text{effective,weekly}}\right]^4
\]
\[
1 + r_{\text{effective,monthly}} = \left[1 + r_{\text{effective,weekly}}\right]^{28} - 1 \equiv 0.00249 = 0.249\%
\]

Effective to Effective: n<1

The effective annual rate is 12%, what is the effective rate for 10 months?
Since the effective annual rate is known and there are \((10/12)=0.8333\) years in a period of 10 months we get
\[
r_{\text{effective,annual}} = 10\% = 0.1
\]
\[
1 + r_{\text{effective,10-months}} = \left[1 + r_{\text{effective,12-months}}\right]^{0.8333}
\]
\[
1 + r_{\text{effective,10-months}} = [1 + 0.1]^3 - 1 \equiv 0.0827 = 8.27\%
\]

Quoted to Effective: Example

The offer: a credit card with 9% APR (annual percentage rate), 9% is the quoted (stated) annual interest rate.
The convention: since credit card payments are monthly, the frequency of compounding is monthly or \(m=12\) times in one year.
The terminology: 9% a year, compounded monthly.
The problem: 9% is NOT the effective annual interest rate (9% is not the annual rate of return).
To compare this rate to other offers or discount annual CFs we need the effective rate.

Example Continued

Since the frequency of compounding is monthly, start by finding the effective monthly rate:
\[
r_{\text{quoted,annual}} = r_{\text{quoted,12-month}} = 9\% = 0.09
\]
\[
r_{\text{effective,1-month}} = \frac{r_{\text{quoted,1-month}}}{m}
\]
\[
r_{\text{effective,1-month}} = \frac{r_{\text{quoted,12-month}}}{12}
\]
\[
r_{\text{effective,1-month}} = \frac{0.09}{12} = 0.0075 = 0.75\%
\]

Example Continued

Now, use the effective-to-effective formula to find any other effective interest rate.
What is the effective annual interest rate?
(Remember: there are \(n=12\) months in one year)
\[
r_{\text{effective,monthly}} = 0.0075 = 0.75\%
\]
\[
1 + r_{\text{effective,monthly}} = \left[1 + r_{\text{effective,monthly}}\right]^{12}
\]
\[
1 + r_{\text{effective,monthly}} = [1 + 0.0075]^2 - 1 = 1.0938
\]
\[
r_{\text{effective,annual}} = 1.0938 - 1 = 0.0938 = 9.38\%
\]

Example Continued

What is the effective quarterly interest rate? (Remember: there are \(n=3\) months in one quarter)
\[
r_{\text{effective,monthly}} = 0.0075 = 0.75\%
\]
\[
1 + r_{\text{effective,monthly}} = \left[1 + r_{\text{effective,monthly}}\right]^3
\]
\[
1 + r_{\text{effective,monthly}} = [1 + 0.0075]^3 - 1 = 1.0227
\]
\[
r_{\text{effective,annual}} = 1.0227 - 1 = 0.0227 = 2.27\%
\]
Example Continued

What is the effective weekly interest rate?
(Remember: there are 4 weeks in one month or \( n = 1/4 = 0.25 \) months in one week)

\[
\begin{align*}
1 + r_{\text{effective, weekly}} &= \left(1 + r_{\text{effective, monthly}}\right)^{0.25} \\
1 + r_{\text{effective, weekly}} &= \left(1 + 0.0075\right)^{0.25} \approx 1.0019 \\
r_{\text{effective, annual}} &= 1.0019 - 1 = 0.0019 = 0.19% 
\end{align*}
\]

Quoted to Effective: Formula

\[
r_{\text{quoted, m-period}} \approx \text{quoted rate, compounded m times} \\
r_{\text{effective, 1-period}} \approx \text{1-period effective rate} \\
1 + r_{\text{effective, 1-period}} &= \left(1 + r_{\text{effective, monthly}}\right) \\
1 + r_{\text{effective, 1-period}} &= \left[1 + r_{\text{effective, monthly}}\right]^{0.25} \approx 1.0019 \\
r_{\text{effective, annual}} &= 1.0019 - 1 = 0.0019 = 0.19% 
\]

Quoted to Effective: Example

You plan to buy a car for $45,000. The dealer offers to finance the entire amount and requires 60 monthly payments of $950.

1. What is the effective monthly interest rate?
2. What annual interest rate will the dealer state (quote)?
3. What is the effective annual interest rate?

Quoted to Effective: Example

Your bank states that the interest rate on a three month certificate of deposit (CD) is 4.68% per annum.

1. What is the quoted (stated) interest rate?
2. What is the frequency of compounding?
3. What is the effective annual interest rate?

Quoted to Effective: Example

You are trying to borrow $200,000 to buy a house on a conventional 30-year mortgage with monthly payments. Your bank is asking for 8.4% a year.

1. What is the quoted (stated) interest rate?
2. What is the frequency of compounding?
3. What is the effective annual interest rate?
4. What is the monthly payment on the loan?

Inflation

The inflation rate (\( i \)): the rate of a general rise in prices over time. If \( i > 0 \) then the same commodity becomes more expensive over time.

If you could buy a product for $100 in 2005, in 2006 you had to pay $103.23 for the same product. In 2007 you had to pay $106.17 and in 2008, $110.24. The implied inflation rates are:

<table>
<thead>
<tr>
<th>Date</th>
<th>Price</th>
<th>Inflation rate - ( i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005</td>
<td>$100.00</td>
<td></td>
</tr>
<tr>
<td>2006</td>
<td>$103.23</td>
<td>103.23 / 100.00 - 1 = 3.23%</td>
</tr>
<tr>
<td>2007</td>
<td>$106.17</td>
<td>106.17 / 103.23 - 1 = 2.85%</td>
</tr>
<tr>
<td>2008</td>
<td>$110.24</td>
<td>110.24 / 106.17 - 1 = 3.83%</td>
</tr>
</tbody>
</table>
Interest rates and inflation

The real interest rate \( (r_{\text{real}}) \): the return (compensation) you demand for lending someone money and thus postponing consumption.

The nominal interest rate \( (r_{\text{nominal}}) \): inflation-adjusted interest rate that represents compensation for both: inflation and postponing consumption.

In the real world, all the quoted rates are nominal rates (e.g., car loan, house loan, student loan).

Nominal and real Interest rates

The nominal annual rate of return in 2006 was 4.91% (30 year US-Treasury bond), what is the real annual rate of return?

\[
(1 + r_{\text{nominal}}) = (1 + r_{\text{real}}) \times (1 + i)
\]

\( i \) = inflation rate

\( r_{\text{nominal}} \) = nominal interest rate

\( r_{\text{real}} \) = real interest rate

Note: all rates must be for the same period (say, one year).

Examples

1. If the real interest rate is 8% and the inflation rate is 4%, what is the nominal interest rate?

2. If the nominal interest rate is 12.2% and that inflation rate is 3.6%, what is the real interest rate?

Textbook Example: Annuity due

You own a property that you want to rent for 10 years.

Prospective tenant A promises to pay $12,000 per year with payments made at the end of each year.

Prospective tenant B promises to pay $12,000 per year with payments made at the beginning of each year.

Which is the better deal if the appropriate annual discount rate is 10%?
The Relation between
(ordinary) annuity and annuity due

\[ PV(\text{annuity due}) = PV(\text{ordinary annuity}) \times (1 + r) \]

\[ FV(\text{annuity due}) = FV(\text{ordinary annuity}) \times (1 + r) \]

Textbook example: loan amortization

You borrowed $8,000 from a bank and promised to repay the loan in five equal annual payments. The first payment is at the end of the first year. The annual interest rate is 10%. Write down the amortization schedule for this loan.

Compute the annual payment ($2,110.38)

Textbook example: loan amortization

We separate each payment into two parts:
- Interest payment
- Repayment of principal

For a fixed payment loan:
- Total payment is fixed
- Interest payment decreases over time
- Principal repayment increases over time

Amortization schedule table

<table>
<thead>
<tr>
<th>Date</th>
<th>Beginning Balance</th>
<th>Total Payment</th>
<th>Interest (10%)</th>
<th>Principal</th>
<th>End Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8,000.00</td>
<td></td>
<td></td>
<td></td>
<td>8,000.00</td>
</tr>
<tr>
<td>1</td>
<td>8,000.00</td>
<td>2,110.38</td>
<td>800</td>
<td>1,310.38</td>
<td>6,689.62</td>
</tr>
<tr>
<td>2</td>
<td>6,689.62</td>
<td>2,110.38</td>
<td></td>
<td>1,310.38</td>
<td>6,689.62</td>
</tr>
<tr>
<td>3</td>
<td>6,689.62</td>
<td>2,110.38</td>
<td></td>
<td>1,310.38</td>
<td>6,689.62</td>
</tr>
<tr>
<td>4</td>
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<td>2,110.38</td>
<td></td>
<td>1,310.38</td>
<td>6,689.62</td>
</tr>
<tr>
<td>5</td>
<td>6,689.62</td>
<td>2,110.38</td>
<td></td>
<td>1,310.38</td>
<td>6,689.62</td>
</tr>
</tbody>
</table>

loan amortization: solution

First year:
Beginning balance = 8,000
Interest payment = 8,000 \times 0.1 = 800
Principal repayment = 2,110.38 - 800 = 1,310.38
New principal balance = 8,000 - 1,310.38 = 6,689.62
loan amortization: solution

Second year:
Beginning balance = 6,689.62
Interest payment = 6,689.62 \times 0.1 = 668.96
Principal repayment = 2,110.38 - 668.96 = 1,441.42
New principal balance = 6,689.62 - 1,441.42 = 5,248.20

Amortization schedule

<table>
<thead>
<tr>
<th>Date</th>
<th>Beginning Balance</th>
<th>Total Payment</th>
<th>Interest (10%)</th>
<th>Principal</th>
<th>End Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8,000.00</td>
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<td></td>
<td>8,000.00</td>
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<td>800.00</td>
<td>1,310.38</td>
<td>6,689.62</td>
</tr>
<tr>
<td>2</td>
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<td>191.85</td>
<td>1,918.53</td>
<td>0.00</td>
</tr>
<tr>
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<td>191.85</td>
<td>1,918.53</td>
<td>0.00</td>
</tr>
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<td>2,110.38</td>
<td>191.85</td>
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