A Call Option

A European call option gives the buyer of the option the right to purchase the underlying asset, at the exercise price on expiration date.

It is optimal to exercise the call option if the stock price exceeds the strike price:

\[ C_T = \max\{S_T - X, 0\} \]

Buying a Call – Payoff Diagram

<table>
<thead>
<tr>
<th>Stock price ( S_T )</th>
<th>Payoff = Max{S_T-X, 0}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
</tr>
</tbody>
</table>

A Put Option

A European put option gives the buyer of the option the right to sell the underlying asset, at the exercise price on expiration date.

It is optimal to exercise the put option if the stock price is below the strike price:

\[ P_T = \max\{X - S_T, 0\} \]

Buying a Put – Payoff Diagram

<table>
<thead>
<tr>
<th>Stock price ( S_T )</th>
<th>Payoff = Max{X-S_T, 0}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>25</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>0</td>
</tr>
</tbody>
</table>
The Put Call Parity

Compare the payoffs of the following strategies:

- **Strategy I:**
  - Buy one call option (strike= X, expiration= T)
  - Buy one risk-free bond (face value= X, maturity= T, return= rf)

- **Strategy II:**
  - Buy one share of stock
  - Buy one put option (strike= X, expiration= T)

---

Strategy I – Portfolio Payoff

<table>
<thead>
<tr>
<th>Stock price</th>
<th>Buy Call</th>
<th>Buy Bond</th>
<th>All (Portfolio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>25</td>
<td>5</td>
<td>20</td>
<td>25</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>

The Put Call Parity

If two portfolios have the same payoffs in every possible state and date in the future, their prices must be equal:

\[ C + PV(X) = S + P \]

---

Strategy II – Portfolio Payoff

<table>
<thead>
<tr>
<th>Stock price</th>
<th>Buy Stock</th>
<th>Buy Put</th>
<th>All (Portfolio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>25</td>
<td>25</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
<td>0</td>
<td>30</td>
</tr>
</tbody>
</table>

Arbitrage – the Law of One Price

If two assets have the same payoffs in every possible state in the future and their prices are not equal, there is an opportunity to make arbitrage profits.

We say that there exists an arbitrage opportunity if we identify that:

- There is no initial investment
- There is no risk of loss
- There is a positive probability of profit

Arbitrage – a Technical Definition

Let \( CF_{ij} \) be the cash flow of an investment strategy at time \( t \) and state \( j \). If the following conditions are met this strategy generates an arbitrage profit:

1. All the possible cash flows in every possible state and time are positive or zero: \( CF_{ij} \geq 0 \) for every \( t \) and \( j \).
2. At least one cash flow is strictly positive: there exists a pair \( (t, j) \) for which \( CF_{ij} > 0 \).
Arbitrage Example
Is there an arbitrage opportunity if we observe the following market prices:
The price of one share of stock is $39;
The price of a call option on that stock, which expires in one year and has an exercise price of $40, is $7.25;
The price of a put option on that stock, which expires in one year and has an exercise price of $40, is $6.50;
The annual risk free rate is 6%.

Arbitrage Example
In this case we must check whether the put call parity holds. Since we can see that this parity relation is violated, we will show that there is an arbitrage opportunity.
\[ C + \frac{X}{(1 + rf)^T} = \frac{7.25}{(1 + 0.06)^1} + \frac{40}{(1 + 0.06)^1} = 44.986 \]
\[ S + P = 39 + 6.50 = 45.5 \]

Construction of an Arbitrage Transaction
Constructing the arbitrage strategy:
1. Move all the terms to one side of the equation so their sum will be positive;
2. For each asset, use the sign as an indicator of the appropriate investment in the asset:
   - If the sign is negative then the cash flow at time \( t=0 \) is negative (which means that you buy the stock, bond or option).
   - If the sign is positive reverse the position.

Arbitrage Example
In this case we move all terms to the LHS:
\[ (S + P)(C + \frac{X}{(1 + rf)^T}) = 45.5 - 44.986 = 0.514 > 0 \]
i.e.
\[ S + P - C - \frac{X}{(1 + rf)^T} > 0 \]

Arbitrage Example
In this case we should:
1. Sell (short) one share of stock
2. Write one put option
3. Buy one call option
4. Buy a zero coupon risk-free bond (lend)
Arbitrage Example

<table>
<thead>
<tr>
<th>Time:</th>
<th>t = 0</th>
<th>t = T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy: ↓</td>
<td>State: ↑</td>
<td>S_T &lt; X = 40</td>
</tr>
<tr>
<td>Short stock</td>
<td>+S = $39</td>
<td>-S_T</td>
</tr>
<tr>
<td>Write put</td>
<td>+P = $6.5</td>
<td>-S_T</td>
</tr>
<tr>
<td>Buy call</td>
<td>-C = ($7.25)</td>
<td>0</td>
</tr>
<tr>
<td>Buy bond</td>
<td>-X(1+rf) = (-$37.736)</td>
<td>X</td>
</tr>
<tr>
<td>Total CF</td>
<td>S + P - C - X(1+rf) = 0.514</td>
<td>0</td>
</tr>
</tbody>
</table>

Arbitrage Example Continued

Is there an arbitrage opportunity if we observe the following market prices:

- The price of one share of stock is $37;
- The price of a call option on that stock, which expires in one year and has an exercise price of $40, is $7.25;
- The price of a put option on that stock, which expires in one year and has an exercise price of $40, is $6.50;
- The annual risk free rate is 6%.

In this case the put call parity relation is violated again, and there is an arbitrage profit opportunity.

\[
C + \frac{X}{(1+rf)^T} = 7.25 + \frac{40}{(1+0.06)^T} = 44.986 \\
S + P = 37 + 6.50 = 43.5
\]

Arbitrage Example Continued

In this case we get

\[
\left(C + \frac{X}{(1+rf)^T}\right) - (S + P) = 44.986 - 43.5 = 1.486 > 0
\]

i.e.

\[
C + \frac{X}{(1+rf)^T} - S - P > 0
\]
The Value of a Call Option

**Assumptions:**
1. A European Call option
2. The underlying asset is a stock that pays no dividends before expiration
3. The stock is traded
4. A risk free bond is traded

**Arbitrage restrictions:**
\[ \max \{ S - PV(X), 0 \} < C_{EU} < S \]

**Example:**
The current stock price is $83
- The stock will not pay dividends in the next six months
- A call option on that stock is traded for $3
- The exercise price is $80
- The expiration of the option is in 6 months
- The 6 months risk free rate is 5%

Is there an opportunity to make an arbitrage profit?

**Example Continued**

<table>
<thead>
<tr>
<th>Time: ( t )</th>
<th>( t = 0 )</th>
<th>( t = T )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Strategy:</strong></td>
<td>↓</td>
<td></td>
</tr>
<tr>
<td><strong>State:</strong></td>
<td>( S_T &lt; X = 80 )</td>
<td>( S_T \geq X = 80 )</td>
</tr>
<tr>
<td>Short stock</td>
<td>+S=$83</td>
<td>-S_T</td>
</tr>
<tr>
<td>Buy call</td>
<td>-C=(-$3)</td>
<td>0</td>
</tr>
<tr>
<td>Buy bond</td>
<td>-X/(1+rf)=(56.19)</td>
<td>+X</td>
</tr>
<tr>
<td>Total CF</td>
<td>+S-C-X/(1+rf)</td>
<td>-S_T+X &gt; 0</td>
</tr>
</tbody>
</table>

The call option price is bounded
\[ \max \{ S - PV(X), 0 \} < C_{EU} < S \]

The call option price is monotonically increasing in the stock price \( S \)
If \( S \uparrow \) then \( C(S) \uparrow \)

The call option price is a convex function of the stock price \( S \) (see sketch).
The Value of a Call Option

Exercise Prices

Assume there are two European call options on the same stock \( S \), with the same expiration date \( T \), that have different exercise prices \( X_1 < X_2 \).

Then,

\[
C(X_1) > C(X_2)
\]

I.e., the price of the call option is monotonically decreasing in the exercise price (if \( X \uparrow \) then \( C(X) \downarrow \)).

Example

Show that if there are two call options on the same stock (that pays no dividends), and both have the same expiration date but different exercise prices as follows, there is an opportunity to make arbitrage profits.

\[
X_1 = $40 \quad \text{and} \quad X_2 = $50
\]

\[
C_1 = $3 \quad \text{and} \quad C_2 = $4
\]

Example Continued

<table>
<thead>
<tr>
<th>Time: ( t \rightarrow )</th>
<th>( t = 0 )</th>
<th>( t = T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy: ( \downarrow )</td>
<td>( S_t &lt; X_t = 40 )</td>
<td>( S_T &gt; X_T = 50 )</td>
</tr>
<tr>
<td>State: ( \rightarrow )</td>
<td>( S_T &lt; X_T &lt; X_2 )</td>
<td>( S_T &gt; X_2 = 50 )</td>
</tr>
<tr>
<td>Buy Call 1</td>
<td>( -C_1 = (-$3) )</td>
<td>0</td>
</tr>
<tr>
<td>Sell Call 2</td>
<td>( +C_2 = $4 )</td>
<td>0</td>
</tr>
<tr>
<td>Total CF</td>
<td>( -C_1 + C_2 = $1 )</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
X_1 = $40, \quad X_2 = $50, \quad X_3 = $60
\]

\[
C_1 = $4.6, \quad C_2 = $4, \quad C_3 = $3
\]

Option Price Convexity

Assume there are three European call options on the same stock \( S \), with the same expiration date \( T \), that have different exercise prices \( X_1 < X_2 < X_3 \).

If

\[
X_2 = \alpha X_1 + (1- \alpha) X_3
\]

Then

\[
C(X_2) < \alpha C(X_1) + (1- \alpha)C(X_3)
\]

I.e., the price of a call option is a convex function of the exercise price.

Example

Show that if there are three call options on the same stock (that pays no dividends), and all three have the same expiration date but different exercise prices as follows, there is an opportunity to make arbitrage profits.

\[
X_1 = $40, \quad X_2 = $50 \quad \text{and} \quad X_3 = $60
\]

\[
C_1 = $4.6, \quad C_2 = $4 \quad \text{and} \quad C_3 = $3
\]
Example Continued

<table>
<thead>
<tr>
<th>Time: →</th>
<th>t = 0</th>
<th>t = T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy:</td>
<td>S_T &lt; X_1 = 40</td>
<td>X_1 &lt; S_T &lt; X_2</td>
</tr>
</tbody>
</table>

Buy Call 1 (one unit)

- C_1 = 4.6
- (S_T - X_1)
- (S_T - X_2)

Sell Call 2 (two units)

- C_2 = 8
- 0
- 0
- -2*(S_T - X_2)
- -2*(S_T - X_3)

Buy Call 3 (one unit)

- C_3 = -3
- 0
- 0
- 0
- (S_T - X_3)

Total CF

C_1 + 2C_2 + C_3 = 0.4 > 0

Time:

- t = 0
- t = T
- t = T

The Value of a Call Option

- The call option price is bounded
  \( \max\{ S - PV(X), 0 \} < C_{EU} < S \)
- The Call option price is monotonically decreasing in the exercise price X
  \( X \uparrow \text{ then } C(X) \downarrow \)
- The call option price is a convex function of the exercise price X.

The Value of a Call Option

\[
C
\begin{cases}
0, & X < S - PV(X) \\
S - PV(X), & S - PV(X) \leq X \leq S \\
S, & X > S
\end{cases}
\]

The Value of a Call Option

Assumptions:
1. Two Call options – European and American
2. The underlying asset is a stock that pays no dividends before expiration
3. The stock is traded
4. A risk free bond is traded

Arbitrage restriction:
\( C_{\text{European}} = C_{\text{American}} \)

Hint: compare the payoff from immediate exercise to the lower bound of the European call option price. If \( C_{EU} < C_{AM} \) then you can make arbitrage profits, but the strategy is dynamic and involves transactions in the present and in a future date \( t < T \).

Example

There are two call options on the same stock (that pays no dividends), one is American and one is European. Both have the same expiration date (a year from now, \( T = 2 \)) and exercise price \( X = 100 \) but the American option costs more than the European \( (C_{EU} = 5 < 6 = C_{AM}) \).

Assume that the buyer of the American call option considers to exercise after 6 months \( (t = 1) \). Show that if the semi-annual interest rate is \( rf = 5\% \) then there is an opportunity to make arbitrage profits.
Example Continued

Since the no-arbitrage restriction is $C_{EU} = C_{AM}$ but the market prices are $C_{EU} = 5 < 6 = C_{AM}$, we can make arbitrage profits if we buy the cheap option ($C_{EU} = 5$) and sell (write) the expensive one ($C_{AM} = 6$).

If the buyer of the American call option decides to exercise before expiration (date $t < T$), we should remember that $C_{AM} = S - X < S - PV(X) < C_{EU}$ and (1) sell the stock, (2) buy a bond.

If the buyer of the American call option decides to exercise only on expiration date, then the future CFs of the American and European call options will offset each other.

If the American Call is not Exercised before Expiration

<table>
<thead>
<tr>
<th>Time: $t$</th>
<th>$t = 0$</th>
<th>$t = 2 = T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy:</td>
<td>$S_T &lt; X = 100$</td>
<td>$S_T &gt; X = 100$</td>
</tr>
<tr>
<td>Buy Eu Call (date $t=0$)</td>
<td>$C_{EU} = 5$</td>
<td>$0$</td>
</tr>
<tr>
<td>Sell Am Call (date $t=0$)</td>
<td>$C_{AM} = 6$</td>
<td>$(S_T - X)$</td>
</tr>
<tr>
<td>Total CF</td>
<td>$C_{AM} - C_{EU} = 6 - 5 &gt; 0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Application

Say only a European option is traded in the market and on date $t=1$ you really wish you could exercise since the price is much lower than the strike (say $S_T = 150$).

Describe a strategy that will be as good as (if not better than) exercising a call option before expiration.

Intuition – short stock and long bond will generate at least the same payoff as exercising an American call option. We can use this strategy to “exercise” the European call option.

Note: if you own an American call option, the same intuition implies that you should guarantee the profit rather than exercise. The payoff of adding short stock and long bond to your American call is better than exercising before expiration.

The Value of a Put Option

Assumptions:
1. A European put option
2. The underlying asset is a stock that pays no dividends before expiration
3. The stock is traded
4. A risk free bond is traded

Arbitrage restrictions:
$$\text{Max}\{PV(X) - S, 0\} < P_{EU} < PV(X)$$
The Value of a Put Option

Max\{ PV(X) - S, 0 \} < P_{EU} < PV(X)

- 0 < P: the owner has a right but not an obligation.
- PV(X) - S < P: arbitrage proof.
- P < PV(X): the highest profit from the put option is realized when the stock price is zero. That profit is $X and it is realized on date T, which is equivalent to PV($X) today.

Example

The current stock price is $75
The stock will not pay dividends in the next six months
A put option on that stock is traded for $1
The exercise price is $80
The expiration of the option is in 6 months
The 6 months risk free rate is 5%
Is there an opportunity to make arbitrage profits?

Example Continued

<table>
<thead>
<tr>
<th>Time: t = 0</th>
<th>t = T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy: ↓</td>
<td>State: →</td>
</tr>
</tbody>
</table>
| Buy stock | S_T < X = 80  
- S_T = -($75)  
+ S_T |
| Buy put | (X - S_T)  
- P = (-$1) |
| Sell bond | + X/(1+r)^T = $76.19  
- X |
| Total CF | -S - P + X/(1+r)^T = $0.19 > 0  
+ S_T + (X - S_T) - X = 0  
S_T - X > 0 |

The put price is bounded
Max\{ PV(X) - S, 0 \} < P_{EU} < PV(X)

- The put option price is monotonically decreasing in the stock price S
  If S↑ then P(S)↓
- The put option price is a convex function of the stock price S (see sketch).
Exercise Prices

Assume there are two European put options on the same stock $S$, with the same expiration date $T$, that have different exercise prices $X_1 < X_2$.

Then,

$$P(X_1) < P(X_2)$$

i.e., the price of the put option is monotonically increasing in the exercise price (if $X \uparrow$ then $P(X) \uparrow$).

Example

Show that if there are two put options on the same stock (that pays no dividends), and both have the same expiration date but different exercise prices as follows, there is an opportunity to make arbitrage profits.

$X_1 = 40$ and $X_2 = 50$  
$P_1 = 4$ and $P_2 = 3$

Example Continued

<table>
<thead>
<tr>
<th>Time: $\rightarrow$</th>
<th>t = 0</th>
<th>t = T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy: $\downarrow$</td>
<td>State: $\rightarrow$</td>
<td>$S_1 &lt; X_1 = 40$</td>
</tr>
<tr>
<td>Sell Put 1 ($X_1 = 40$)</td>
<td>$P_1 = 4$</td>
<td>$- (X_1 - S_1)$</td>
</tr>
<tr>
<td>Buy Put 2 ($X_2 = 50$)</td>
<td>$- P_2 = (-3)$</td>
<td>$(X_2 - S_1)$</td>
</tr>
<tr>
<td>Total CF</td>
<td>$P_1 - P_2$</td>
<td>$X_2 - X_1$</td>
</tr>
</tbody>
</table>
| | = $1 > 0$ | $50 - 40 > 0$ | | $= 0$

Option Price Convexity

Assume there are three European put options on the same stock $S$, with the same expiration date $T$, that have different exercise prices $X_1 < X_2 < X_3$.

If $X_2 = \alpha X_1 + (1 - \alpha) X_3$

Then

$$P(X_2) < \alpha P(X_1) + (1 - \alpha)P(X_3)$$

i.e., the price of a put option is a convex function of the exercise price.

Example

Show that if there are three put options on the same stock (that pays no dividends), and all three have the same expiration date but different exercise prices as follows, there is an opportunity to make arbitrage profits.

$X_1 = 40$, $X_2 = 50$ and $X_3 = 60$  
$P_1 = 3$, $P_2 = 4$ and $P_3 = 4.6$

Example Continued

<table>
<thead>
<tr>
<th>Time: $\rightarrow$</th>
<th>t = 0</th>
<th>t = T</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategy: $\downarrow$</td>
<td>State: $\rightarrow$</td>
<td>$S_1 &lt; X_1 = 40$</td>
</tr>
<tr>
<td>Buy put 1 (one unit)</td>
<td>$-P_1 = -3$</td>
<td>$(X_1 - S_1)$</td>
</tr>
<tr>
<td>Sell put 2 (two units)</td>
<td>$2P_2 = 8$</td>
<td>$-2(X_2 - S_1)$</td>
</tr>
<tr>
<td>Buy put 3 (one unit)</td>
<td>$-P_3 = -4.6$</td>
<td>$(X_2 - S_1)$</td>
</tr>
<tr>
<td>Total CF</td>
<td>$P_1 + 2P_2 + P_3$</td>
<td>$X_1 + X_1 - 2X_2 = 40 -60 -100$</td>
</tr>
</tbody>
</table>
The Value of a Put Option

- The put price is bounded
  \[ \text{Max}\{ \text{PV}(X)-S, 0 \} < P_{EU} < \text{PV}(X) \]
- The put option price is monotonically increasing in the exercise price \( X \)
  If \( X \uparrow \) then \( P(X) \uparrow \)
- The put option price is a convex function of the exercise price \( X \).

The Value of a Put Option

Assumptions:
1. Two Put options — European and American
2. The underlying asset is a stock that pays no dividends before expiration
3. The stock is traded
4. A risk free bond is traded

For a put option, regardless of dividends:
\[ P(\text{European}) \leq P(\text{American}) \]
Note: in this case an example is enough.

Determinants of the Values of Call and Put Options

<table>
<thead>
<tr>
<th>Variable</th>
<th>C – Call Value</th>
<th>P – Put Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S ) — stock price</td>
<td>Increase</td>
<td>Decrease</td>
</tr>
<tr>
<td>( X ) — exercise price</td>
<td>Decrease</td>
<td>Increase</td>
</tr>
<tr>
<td>( \sigma ) — stock price volatility</td>
<td>Increase</td>
<td>Increase</td>
</tr>
<tr>
<td>( T ) — time to expiration</td>
<td>Increase</td>
<td>Increase</td>
</tr>
<tr>
<td>( r ) — risk-free interest rate</td>
<td>Increase</td>
<td>Decrease</td>
</tr>
<tr>
<td>( \text{Div} ) — dividend payouts</td>
<td>Decrease</td>
<td>Increase</td>
</tr>
</tbody>
</table>

Practice Problems

BKM Ch. 21:
7th Ed.: End of chapter - 1, 2
Example 21.1 and concept check Q #4
8th Ed.: End of chapter - 1, 6
Example 21.1 and concept check Q #4
Practice set: 17-24