Valuation of Options

- Arbitrage Restrictions on the Values of Options
- Quantitative Pricing Models
  - Binomial model
    - A formula in the simple case
    - An algorithm in the general case
  - Black-Scholes model (a formula)

Binomial Option Pricing Model

Assumptions:
- A single period
  - Two dates: time $t=0$ and time $t=1$ (expiration)
- The future (time 1) stock price has only two possible values
  - The price can go up or down
- The perfect market assumptions
  - No transactions costs, borrowing and lending at the risk free interest rate, no taxes...

Example

The stock price
Assume $S=50$, $u=10\%$ and $d=(-3\%)$

$$S_u = S \cdot (1+u) = S \cdot 1.10$$

$$S_d = S \cdot (1+d) = S \cdot 0.97$$

The bond price
Assume $r=6\%$

$$B = \frac{1}{(1+r)^T} = \frac{1}{1.06}$$

$B = \frac{1}{1.06} = 0.9434$
Replicating Portfolio

At time \( t=0 \), we can create a portfolio of \( N \) shares of the stock and an investment of \( B \) dollars in the risk-free bond. The payoff of the portfolio will replicate the \( t=1 \) payoffs of the call option:

\[
N \cdot \$55 + B \cdot \$1.06 = \$5
\]

\[
N \cdot \$48.5 + B \cdot \$1.06 = \$0
\]

Obviously, this portfolio should also have the same price as the call option at \( t=0 \):

\[
N \cdot \$50 + B \cdot \$1 = C
\]

We get \( N=0.7692, B=-35.1959 \) and the call option price is \( C=\$3.2656 \).

The weights of Bonds and Stocks in the replicating portfolio are:

\[
Z_B = \frac{-35.1959}{3.2656} = -10.78
\]

\[
Z_S = \frac{0.7692 \cdot \$50}{3.2656} = 11.78
\]

Say you invest \$100. The two equivalent investment strategies are:

1. Buy call options for \$100 (i.e., buy \$100 / \$3.2656 = 30.62 call options)
2. Sell bonds for \( 10.78 \cdot \$100 = \$1,078 \)
   Buy stocks for \( 11.78 \cdot \$100 = \$1,178 \)

A Different Replication

The price of \$1 in the "up" state:

\[
q_u = \frac{\$43.54}{\$43.54 + \$0} = 0.4354
\]

The price of \$1 in the "down" state:

\[
q_d = \frac{\$0}{\$43.54 + \$0} = 0.5646
\]

Binomial Option Pricing Model

Example Continued

The put option price Assume \( X= \$50 \), \( T= 1 \) year (expiration)

\[
P_u = \max(\$50 - \$55, 0) = 0
\]

\[
P_d = \max(\$50 - \$48.5, 0) = \$1.5
\]

The put option price is \( P=\$0.4354 \).

The price of \$1 in the "up" state:

\[
q_u = \frac{\$43.54}{\$43.54 + \$0} = 0.4354
\]

The price of \$1 in the "down" state:

\[
q_d = \frac{\$0}{\$43.54 + \$0} = 0.5646
\]
Replicating Portfolios Using the State Prices
We can replicate the $t=1$ payoffs of the stock and the bond using the state prices:
\[ q_u \cdot 55 + q_d \cdot 48.5 = 50 \]
\[ q_u \cdot 1.06 + q_d \cdot 1.06 = 1 \]
Obviously, once we solve for the two state prices we can price any other asset in that economy. In particular we can price the call option:
\[ q_u \cdot 5 \cdot q_d \cdot 0 = C \]
We get \( q_u = 0.6531, \ q_d = 0.2903 \) and the call option price is \( C = 3.2656 \).

Replicating Portfolios Using the State Prices
We can replicate the $t=1$ payoffs of the stock and the bond using the state prices:
\[ q_u \cdot 55 + q_d \cdot 48.5 = 50 \]
\[ q_u \cdot 1.06 + q_d \cdot 1.06 = 1 \]
But the assets are exactly the same and so are the state prices. The put option price is:
\[ q_u \cdot 0 \cdot q_d \cdot 1.5 = P \]
We get \( q_u = 0.6531, \ q_d = 0.2903 \) and the put option price is \( P = 0.4354 \).

Binomial Option Pricing Model
The put option price
\[ \begin{align*}
P_u &= \text{Max}(50 - 55, 0) \\
P_d &= \text{Max}(50 - 48.5, 0) \\
P_u &= 0.2903 \\
P_d &= 0.6531 \\
P &= 1.5 \\
C &= 3.2656 \\
\end{align*} \]

Two Period Example
Assume that the current stock price is $50, and it can either go up 10% or down 3% in each period.

The one period risk-free interest rate is 6%.
What is the price of a European call option on that stock, with an exercise price of $50 and expiration in two periods?

The Stock Price
\[ S = 50, \ u = 10\% \text{ and } d = (-3\%) \]

The Bond Price
\[ r = 6\% \text{ (for each period)} \]
The Call Option Price

X = $50 and T = 2 periods

\[ C_u = \max(60.5 - 50, 0) = 10.5 \]
\[ C_d = \max(53.35 - 50, 0) = 3.35 \]
\[ C_{uu} = \max(47.05 - 50, 0) = 0 \]

State Prices in the Two Period Tree

We can replicate the t=1 payoffs of the stock and the bond using the state prices:

\[ q_u \cdot 55 + q_d \cdot 48.5 = 50 \]
\[ q_u \cdot 1.06 + q_d \cdot 1.06 = 1 \]

Note that if \( u, d \) and \( r \) are the same, our solution for the state prices will not change (regardless of the price levels of the stock and the bond):

\[ q_u \cdot (1+r)^t + q_d \cdot (1+r)^t = (1+r)^t \]

Therefore, we can use the same state-prices in every part of the tree.

The Call Option Price

\[ q_u = 0.6531 \] and \( q_d = 0.2903 \)

\[ C_u = 0.6531 \cdot 10.5 + 0.2903 \cdot 0 = 7.83 \]
\[ C_d = 0.6531 \cdot 3.35 + 0.2903 \cdot 2.19 = 2.19 \]
\[ C = 0.6531 \cdot 7.83 + 0.2903 \cdot 2.19 = 5.75 \]

Two Period Example

- What is the price of a European put option on that stock, with an exercise price of $50 and expiration in two periods?
- What is the price of an American call option on that stock, with an exercise price of $50 and expiration in two periods?
- What is the price of an American put option on that stock, with an exercise price of $50 and expiration in two periods?

The European Put Option Price

\[ q_u = 0.6531 \] and \( q_d = 0.2903 \)

\[ P_u = \max(0) = 0 \]
\[ P_d = \max(2.955) = 2.955 \]
\[ P = 0.6531 \cdot 0 + 0.2903 \cdot 2.955 = 0.858 \]
\[ P_{uu} = 0.6531 \cdot 0 + 0.2903 \cdot 0.858 = 0.25 \]
The European Put Option Price

Another way to calculate the European put option price is to use the put-call-parity restriction:

\[ C + PV(X) = S + P \]

\[ P = C + PV(X) - S \]

\[ = 5.75 + \frac{50}{(1 + .06)} - 50 \]

\[ = \$0.25 \]

The American Put Option Price

\[ q_u = 0.6531 \text{ and } q_d = 0.2903 \]

American Put Option

Note that at time \( t=1 \) the option buyer will decide whether to exercise the option or keep it till expiration.

If the payoff from immediate exercise is higher than the value of keeping the option for one more period (“European”), then the optimal strategy is to exercise:

\[ \text{If } \max( X-S, 0 ) > P \text{ (“European”) } \Rightarrow \text{Exercise} \]

Determinants of the Values of Call and Put Options

<table>
<thead>
<tr>
<th>Variable</th>
<th>( C ) – Call Value</th>
<th>( P ) – Put Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S ) – stock price</td>
<td>Increase</td>
<td>Decrease</td>
</tr>
<tr>
<td>( X ) – exercise price</td>
<td>Decrease</td>
<td>Increase</td>
</tr>
<tr>
<td>( \sigma ) – stock price volatility</td>
<td>Increase</td>
<td>Increase</td>
</tr>
<tr>
<td>( T ) – time to expiration</td>
<td>Increase</td>
<td>Increase</td>
</tr>
<tr>
<td>( r ) – risk-free interest rate</td>
<td>Increase</td>
<td>Decrease</td>
</tr>
<tr>
<td>( \text{Div} ) – dividend payouts</td>
<td>Decrease</td>
<td>Increase</td>
</tr>
</tbody>
</table>

Black-Scholes Model

- Developed around 1970
- Closed form, analytical pricing model
  - An equation
  - Can be calculated easily and quickly (using a computer or even a calculator)
  - The limit of the binomial model if we are making the number of periods infinitely large and every period very small – continuous time
- Crucial assumptions
  - The risk free interest rate and the stock price volatility are constant over the life of the option.
Black-Scholes Model

\[ C = S \cdot N(d_1) - X \cdot e^{-rT} \cdot N(d_2) \]

Where

\[ d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(T + \frac{1}{2}\right) \sigma^2}{\sigma \sqrt{T}} \]

\[ d_2 = d_1 - \sigma \sqrt{T} \]

C – call premium
S – stock price
X – exercise price
T – time to expiration
\( r \) – the interest rate
\( \sigma \) – std of stock returns
\( \ln(z) \) – natural log of \( z \)
\( e^{rT} \) – \( \exp\{rT\} = (2.7183)^{rT} \)
\( N(z) \) – standard normal cumulative probability

Black-Scholes Example

\[ C = S \cdot N(d_1) - X \cdot e^{-rT} \cdot N(d_2) \]

Where

\[ d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(T + \frac{1}{2}\right) \sigma^2}{\sigma \sqrt{T}} \]

\[ d_2 = d_1 - \sigma \sqrt{T} \]

\( S \) – $47.50
\( X \) – $50
\( T \) – 0.25 years
\( r \) – 0.05 (5% annual rate compounded continuously)
\( \sigma \) – 0.30 (or 30%)

The N(0,1) Distribution

\[ \text{pdf}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \]

Black-Scholes Model

- Continuous time and therefore continuous compounding
- \( N(d) \) – loosely speaking, \( N(d) \) is the “risk adjusted” probability that the call option will expire in the money (check the pricing for the extreme cases: 0 and 1)
- \( \ln(S/X) \) – approximately, a percentage measure of option “moneyness”
Black-Scholes Example

\[
d_1 = (-0.1836) \\
N(d_1) = 0.4272 \\
d_2 = (-0.3336) \\
N(d_2) = 0.3693 \\
\text{and} \quad \sigma = 0.30 \text{ (or 30%)} \\
P = $3.9315
\]

P – ?
S – $47.50
X – $50
T – 0.25 years
r – 0.05 (5% annual rate compounded continuously)

The Put Call Parity

The continuous time version (continuous compounding):

\[
C + PV(X) = S + P \\
C + Xe^{-rT} = S + P
\]

\[
\text{and} \quad d_1 = \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{T}
\]

\[
$2.0526 + $50e^{-0.05 \times 0.25} \text{ ? } $47.5 + $3.9315
\]

Stock Return Volatility

One approach:

Calculate an estimate of the volatility using the historical stock returns and plug it in the option formula to get pricing

\[
\text{Est}(\sigma) = \frac{1}{\sqrt{n-1}} \sum_{i=1}^{n} (\text{return}_i - \text{average return})^2
\]

Where

\[
\text{return}_i = \ln \left( \frac{S_i}{S_{i-1}} \right)
\]

Stock Return Volatility

Another approach:

Calculate the stock return volatility implied by the option price observed in the market (a trial and error algorithm)

\[
C = S \cdot N(d_1) - Xe^{-rT} \cdot N(d_2)
\]

Where

\[
d_1 = \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \\
d_2 = d_1 - \sigma \sqrt{T}
\]

Option Price and Volatility

Let \( \sigma_1 < \sigma_2 \) be two possible, yet different return volatilities; \( C_1, C_2 \) be the appropriate call option prices; and \( P_1, P_2 \) be the appropriate put option prices.

We assume that the options are European, on the same stock \( S \) that pays no dividends, with the same expiration date \( T \).

Note that our estimate of the stock return volatility changes. The two different prices are of the same option, and can not exist at the same time!

Then,

\[
C_1 \leq C_2 \text{ and } P_1 \leq P_2
\]

Implied Volatility - example

\[
C = S \cdot N(d_1) - Xe^{-rT} \cdot N(d_2)
\]

Where

\[
d_1 = \frac{\ln \left( \frac{S}{X} \right) + \left( r + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \\
d_2 = d_1 - \sigma \sqrt{T}
\]

\[
C = $2.5 \text{ S } $47.50 \text{ X } $50 \text{ T } 0.25 \text{ years} \quad r = 0.05 \text{ (5% annual rate compounded continuously)}
\]

\[
$2.0526 + $50e^{-0.05 \times 0.25} \text{ ? } $47.5 + $3.9315
\]
Implied Volatility - example

\[ d_1 = (-0.1836) \]
\[ N(d_1) = 0.4272 \]
\[ d_2 = (-0.3336) \]
\[ N(d_2) = 0.3693 \]

\[ \text{and} \]
\[ C = 2.0526 < 2.5 \]
\[ \sigma < 0.3 \text{ or } \sigma > 0.3 ? \]

Implied Volatility - example

\[ d_1 = (-0.0940) \]
\[ N(d_1) = 0.4626 \]
\[ d_2 = (-0.2940) \]
\[ N(d_2) = 0.3844 \]

\[ \text{and} \]
\[ C = 2.9911 > 2.5 \]

Application: Portfolio Insurance

Options can be used to guarantee minimum returns from an investment in stocks.

Purchasing portfolio insurance (protective put strategy):
- Long one stock;
- Buy a put option on one stock;
- If no put option exists, use a stock and a bond to replicate the put option payoffs.

Portfolio Insurance Example

You decide to invest in one share of General Pills (GP) stock, which is currently traded for $56. The stock pays no dividends.

You worry that the stock’s price may decline and decide to purchase a European put option on GP’s stock. The put allows you to sell the stock at the end of one year for $50.

If the std of the stock price is \( \sigma = 0.3 \) (30%) and \( r = 0.08 \) (8% compounded continuously), what is the price of the put option?

What is the CF from your strategy at time \( t=0 \)?

What is the CF at time \( t=1 \) as a function of \( 0 < S_T < 100 \)?
Portfolio Insurance Example

The B&S formula for the put option:
\[ P = -X e^{-rT} \Phi(d_2) + S_0 \Phi(d_1) \]

Therefore the insurance strategy (Original portfolio + synthetic put) is:
- Long one share of stock
- Long \( X \cdot \Phi(d_2) \) bonds
- Short \( \Phi(d_1) \) stocks

Portfolio Insurance Example

The total time \( t=0 \) CF of the protective put (insured portfolio) is:
\[ C_F(0) = -S_0 P_0 \]
\[ = -S_0 X e^{-rT} \Phi(d_2) + S_0 \Phi(d_1) \]
\[ = -S_0 \Phi(d_1) X e^{-rT} \Phi(1-N(d_2)) \]

And the proportion invested in the stock is:
\[ w_{stock} = \frac{S_0 \Phi(d_1)}{-S_0 \Phi(d_1) X e^{-rT} \Phi(1-N(d_2))} \]

Finaly, the proportion invested in the bond is:
\[ w_{bond} = 1 - w_{stock} \]

Portfolio Insurance Example

The proportion invested in the stock is:
\[ w_{stock} = \frac{S_0 \Phi(d_1)}{S_0 \Phi(d_1) + X e^{-rT} \Phi(1-N(d_2))} \]

Or, if we remember the original (protective put) strategy:
\[ w_{stock} = \frac{S_0 \Phi(d_1)}{S_0 + P_0} \]

Now you have a \$1,054.27 portfolio

Time \( t=1 \) (beginning of week 2):
\[ w_{stock} = \frac{60 \cdot 0.8476}{(60 \cdot 0.8476) + 2.38} = 82.53\% \]
\[ Stock \ value = 0.8253 \cdot \$1,054.27 = \$870.06 \]
\[ Bond \ value = 0.1747 \cdot \$1,054.27 = \$184.21 \]

I.e. you should rebalance your portfolio (increase the proportion of stocks to 82.53\% and decrease the proportion of bonds to 17.47\%).

Why should we rebalance the portfolio? Should we rebalance the portfolio if we use the protective put strategy with a real put option?

Portfolio Insurance Example

Practice Problems

BKM Ch. 21
7th Ed.: 7-10, 17, 18
8th Ed.: 11-14, 17, 18

Practice set: 36-42.