Regression, Correlation, and the Time Interval: Additive-Multiplicative Framework

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When two random variables are both additive or multiplicative, the effect of the way one “slices” the available period to subperiods (time intervals) is well documented in the literature. In this paper, we investigate the time interval effect when one of the variables is additive and one is multiplicative. We prove that the squared multiperiod correlation coefficient ($\rho_t^2$) decreases monotonically as $n$ increases, and approaches zero when $n$ goes to infinity. However, for relevant data corresponding to the U.S. stock market index, when shifting from weekly parameters to quarterly parameters the decrease in $\rho_t^2$ is negligible. The effect on the regression coefficient is much more dramatic and even a shift from weekly data to quarterly data affects the regression coefficient substantially. The regression slope generally approaches zero, minus infinity or plus infinity, as the number of periods increases. Mortonicity, however, exists only in certain cases.

(Correlation Coefficient; Regression Coefficient (Beta); Time Interval)

1. Introduction
The association between variables is commonly measured by regression and correlation coefficients. The data for such analyses are sometimes limited, and in such instances researchers frequently employ monthly, weekly, or even daily data to increase the number of observations. However, even when ample data are available, data corresponding to various arbitrary time intervals (monthly, quarterly, etc.) are commonly employed, and in most studies there is no theoretical justification for the selected time intervals. Moreover, most empirical studies do not even address this issue. Thus, in view of the fact that a given set of data can be “sliced” in various ways, the question of whether the time interval selected affects the results is clearly important.

If the regression and the correlation coefficients are unaffected by the selected time interval (or the “slicing” method), the time interval does not constitute an issue at all. However, if these two parameters are systematically affected by the time interval employed, then the time interval cannot be selected arbitrarily. For example, if a decision is based on monthly data when in fact the annual data are relevant, the decision drawn from the statistical results is likely to be incorrect and lead to misguided actions.

If the data contain serial correlation, it is obvious that the selected time interval may affect the results. However, we claim that such effects occur even if all random variables are independent over time, a counterintuitive result. Thus, in the rest of this paper we assume the variables are independent over time. We focus in this paper on studies in finance; however, our results are relevant to studies conducted in other fields where regression or correlation analyses are employed.

Investigation of the effect of the selected time interval on variables is not new. Levy (1972) shows that the Reward to Volatility—Sharpe’s performance index (1966) measuring the performance of mutual funds—
changes systematically and in a predictable way with changes in the time interval. Fund A may outperform Fund B with monthly data, whereas the opposite holds true with annual data. Levhari and Levy (1977) show that the systematic risk (beta) of securities changes with the length of the time interval: Stocks with high risk become even more risky as the time interval increases, whereas the opposite holds regarding stock characterized by relatively low risk. Levy and Schwartz (1997) show that when both the dependent variable and the explanatory variable are multiplicative over time (e.g., rates of return on assets), the correlation decreases monotonically as the time interval increases, approaching zero as the time interval approaches infinity (excluding the homotetic case, where the correlation coefficient equals -1). All these results are theoretical and do not depend on a given set of empirical data.

Note that when rates of return are analyzed, both random variables are multiplicative. If both the dependent and the explanatory variables are additive, it is easy to show that the regression coefficients as well as the correlation coefficients are unaffected by the selected time interval.

In many cases one variable is additive by its very nature and the other is multiplicative (e.g., gross domestic product (GDP) is additive, and population growth or rates of return on assets, multiplicative). To the best of our knowledge, the implication of the time horizon on the regression coefficient and the correlation coefficient in such cases has not been investigated. We provide a few examples taken from the financial literature corresponding to the additive-multiplicative case. However, such instances probably also exist in other fields of research where association between variables is measured (e.g., biology, chemistry, and psychology).

The following studies in the area of finance employ additive and multiplicative variables simultaneously. Chen et al. (CRR) (1986) explain expected rates of return on risky assets by several variables (industrial production, inflation, interest rate, etc.). Some of the variables are multiplicative and some additive. For example, the growth rate in industrial production (multiplicative variable) becomes additive following logarithm transformation, and some variables do not undergo transformation and remain multiplicative (e.g., the return on "Baa and under" bonds less the return on long-term government bonds: in their notation the variable URR(+), see CRR, p. 389). Similarly, the return on the New York Stock Exchange index remains multiplicative by its nature (CRR's variables ElnY and VlnY). Thus, their regression analyses include both multiplicative and additive variables. Elton et al. (1995) use a similar model to explain the expected rate of return in the bond market. Once again, some of the variables employed are additive and some are multiplicative.

Similar time interval effects have also been observed in empirical analyses in the area of financial accounting. For example, Easton and Harris (1991) run a regression in which the dependent variable is the rate of return on asset \( j \) in period \( t \) (\( R_{ij} \) - multiplicative variable), and the explanatory variable is the accounting earnings, \( A_{ij} \), which by its nature is additive.\(^1\)

These few examples demonstrate the importance of analyzing the time interval effect on the regression coefficients in the additive-multiplicative case. Next we turn to the main results of this paper.

2. The Relationship Between the One-Period and \( n \)-Period Correlation

Let \((X_1, Y_1), (X_2, Y_2), \ldots\) be a sequence of independent, identically distributed (i.i.d.) pairs of variables. \( X_t \) and \( Y_t \) may be dependent, but \( X_1, X_2, \ldots, X_n \) are independent over time, and so are \( Y_1, Y_2, \ldots, Y_n \). We define two new variables to denote an \( n \)-fold increase of differencing interval, one multiplicative and the other additive:

The additive variable, denoted by \( W_n \), is given by

\[
W_n = X_1 + X_2 + \cdots + X_n.
\]

\(^1\)Their model is even more complicated because the accounting earnings \( A_{ij} \) are divided by the price at the beginning of the period \( P_{i-1} \). If the firm pays out all of its earnings as cash dividends, the price is expected not to grow; hence, we have a pure additive-multiplicative framework. However, for any realistic dividend payout ratio smaller than 1, \( P_{i-1} \) also is expected to grow in a multiplicative way, which complicates the analysis because it appears in the denominator of the additive variable \( A_{ij} \).
The multiplicative variable, denoted by $V_n$, is given by

$$V_n = Y_1 \cdot Y_2 \cdots \cdot Y_n.$$  

Let the one-period expected value of $X$ and $Y$, respectively, be

$$E(X) = \mu_X \text{ and } E(Y) = \mu_Y.$$  

The one-period variances are denoted by

$$\text{Var}(X) = \sigma_X^2 \text{ and } \text{Var}(Y) = \sigma_Y^2.$$  

The one-period covariance and correlation coefficient are given, respectively, by

$$\text{Cov}(X_t, Y_t) = \sigma_{XY} \text{ and } \rho_1 = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}.$$  

Because $X_t, X_{t+1}, \ldots, X_n$ are i.i.d., we have

$$E(W_n) = E\left(\sum_{t=1}^{n} X_t\right) = \sum_{t=1}^{n} (\mu_X) = n \cdot \mu_X$$  

(1)

and

$$\text{Var}(W_n) = \text{Var}\left(\sum_{t=1}^{n} X_t\right) = \sum_{t=1}^{n} (\sigma_X^2) = n \cdot \sigma_X^2.$$  

(2)

Similarly, because $Y_t, Y_{t+1}, \ldots, Y_n$ are i.i.d., we have

$$E(V_n) = E\left(\prod_{t=1}^{n} Y_t\right) = \prod_{t=1}^{n} (\mu_Y) = \mu_Y^n$$  

(3)

and

$$\text{Var}(V_n) = \text{Var}\left(\prod_{t=1}^{n} Y_t\right) = (\sigma_Y^2 + \mu_Y^2)^n - \mu_Y^{2n}.$$  

(4)

(for the development of the relationship of Equation (4), see Tobin 1965, Levy 1972, Levy and Levhari 1977). As $(X_1, Y_1, (X_2, Y_2), \ldots$ is a sequence of i.i.d. variables, we obtain

$$\text{Cov}(W_n, V_n) = E(W_n \cdot V_n) - E(W_n) \cdot E(V_n)$$  

$$= E\left(\sum_{t=1}^{n} X_t \cdot \prod_{t=1}^{n} Y_t\right) - E\left(\sum_{t=1}^{n} X_t\right) \cdot E\left(\prod_{t=1}^{n} Y_t\right)$$  

$$= E[(X_1 + X_2 + \cdots + X_n) \cdot (Y_1 \cdot Y_2 \cdots \cdot Y_n)]$$  

$$- (n \cdot \mu_X) \cdot (\mu_Y)^n$$  

$$= E[(X_1 \cdot Y_1 \cdot Y_2 \cdots \cdot Y_n) + \cdots + (X_n \cdot Y_1 \cdot Y_2 \cdots \cdot Y_n)]$$  

$$- (n \cdot \mu_X) \cdot (\mu_Y)^n.$$  

Because for every $t \neq s$, $X_t$ and $Y_s$ are independent, we get

$$\text{Cov}(W_n, V_n) = E(X_1 \cdot Y_1) \cdot E(Y_2) \cdots \cdot E(Y_n) + \cdots$$  

$$+ E(X_n \cdot Y_n) \cdot E(Y_1) \cdots \cdot E(Y_{n-1})$$  

$$- n \cdot \mu_X \cdot (\mu_Y)^n$$  

$$= \left[\sum_{t=1}^{n} E(X_t) \cdot E(Y_t)\right] - n \cdot \mu_X \cdot (\mu_Y)^n.$$  

Using the relationship $\sigma_{XY} = E(X \cdot Y) - \mu_X \cdot \mu_Y$ and the fact that $E(X_t \cdot Y_t)$ is the same for all $t = 1, 2, \ldots, n$, we obtain

$$\text{Cov}(W_n, V_n) = n \cdot (\sigma_{XY} + \mu_X \cdot \mu_Y) \cdot (\mu_Y)^n - n \cdot \mu_X \cdot (\mu_Y)^n$$  

$$= n \cdot (\mu_Y)^n \cdot \rho_1 \cdot \sigma_X \sigma_Y.$$  

(5)

Using Equations (2), (4), and (5), the $n$-period correlation coefficient (between $W_n$ and $V_n$), is as follows:

$$\rho_n = \frac{\text{Cov}(W_n, V_n)}{\sigma_{W_1} \cdot \sigma_{V_1}}$$  

$$= \frac{n \cdot \sigma_X \cdot \sigma_Y \cdot \rho_1 \cdot (\mu_Y)^n}{\sqrt{n \cdot \sigma_X^2 \left[(\sigma_Y^2 + \mu_Y^2)^n - (\mu_Y)^{2n}\right]}}.$$  

(6)

Equation (6) provides the relationship between the multiperiod correlation, $\rho_n$, and the one-period correlation, $\rho_1$. It is interesting to see from Equation (6) that $\rho_n$ depends directly on $\mu_Y$ and $\sigma_Y$ (i.e., parameters of the multiplicative variable), but not on the parameters of the additive variable $X$. However, $\rho_n$ depends indirectly on parameters of the additive variable through the one-period correlation, $\rho_1$.

**Theorem 1.** Let $\rho_n$ be the correlation coefficient as defined in (6), then

1. $\rho_n^2$ is monotonically decreasing in $n$

2. $\lim_{n \to \infty} \rho_n = 0$.

**Proof.**

1. We first show that $\rho_n^2$ monotonically decreases in $n$. 

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We will use the definition of the correlation coefficient in Equation (6),

$$\rho_n^2 = \frac{n \cdot \sigma_X^2 \cdot \rho_n^2 \cdot (\mu_X^2)^{n-1}}{(\sigma_Y^2 + \mu_Y^2)^n - (\mu_Y^2)^n}.$$ 

Differentiating \( \rho_n^2 \) with respect to \( n \) we get,

$$\frac{d\rho_n^2}{dn} = \frac{n \cdot (\sigma_Y^2 + \mu_Y^2)^n - (\mu_Y^2)^n}{(\sigma_Y^2 + \mu_Y^2)^n - (\mu_Y^2)^n} \cdot (\sigma_Y^2 + \mu_Y^2)^{n-1} - (\mu_Y^2)^{n-1}.$$ 

To determine whether \( \rho_n^2 \) decreases as \( n \) increases, it is sufficient to show that

$$\frac{d\rho_n^2}{dn} \leq 0.$$ 

Because

$$\{(\sigma_Y^2 + \mu_Y^2)^n - (\mu_Y^2)^n\} \geq 0$$ 

for all positive \( n \), it is sufficient to show that

$$\{(\sigma_Y^2 + \mu_Y^2)^n - (\mu_Y^2)^n\} \geq 0.$$ 

Denoting the last term by \( K \), we will show that for positive \( n \), \( K \leq 0 \).

Let \( X = \ln(\mu_Y^2) - \ln(\mu_Y^2 + \sigma_Y^2) = \ln(\mu_Y^2/(\mu_Y^2 + \sigma_Y^2)) \leq 1 \). It is obvious that \( X = \ln(\mu_Y^2/(\mu_Y^2 + \sigma_Y^2)) \leq 0 \).

We can write the expression denoted by \( K \) in the following form:

$$K = \{(\sigma_Y^2 + \mu_Y^2)^n - (\mu_Y^2)^n\} \cdot (1 + n \cdot X - (\mu_Y^2)^n).$$

Note that \( n \cdot X \leq 0 \) if \( n \cdot X \leq (\sigma_Y^2)^n \). Substituting \( \sigma_Y^2 \) with this expression, we obtain

$$K = \{(\sigma_Y^2 + \mu_Y^2)^n - (\mu_Y^2)^n\} \cdot (1 + n \cdot X - (\mu_Y^2)^n).$$

Because \( (\mu_Y^2)^n \geq 0 \), what is left to show is that \( e^{\sigma_Y^2 \cdot (1 + n \cdot X)} \leq 1 \), or equivalently that

$$f(n \cdot X) = \frac{e^{\sigma_Y^2 \cdot (1 + n \cdot X)}}{1 + n \cdot X} \geq 1.$$
Table 1 The Multiperiod Correlation as a Function of \( \mu_1 \), \( \sigma_1 \), and \( \rho_1 \)

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Note: Parameters: \( \mu_1 = 1.002182 \), \( \sigma_1 = 0.025971 \). These parameters correspond to weekly rates of return on common stock in the United States for the years 1926–1996, see Stocks, Bonds, Bills and Inflation, Ibbotson Association, Chicago 1998 Yearbook.

Because \( \sigma_1^2/(\rho_1^2 \cdot 2 \cdot \mu_1^2) \) is a constant, we conclude that as \( n \) approaches infinity,

\[
\lim_{n \to \infty} \frac{1}{n} \rho_1^2 = \infty,
\]

therefore \( \lim_{n \to \infty} \rho_1 = 0 \), which completes the proof. \( \square \)

Numerical Example. The decrease in \( |\rho_1| \) when \( n \) increases is a function of the selected parameters \( \mu_1 \) and \( \sigma_1 \). Table 1 illustrates the decrease in \( |\rho_1| \) as \( n \) increases for parameters corresponding to the U.S. stock market. We employ weekly data with \( \mu_1 \cong 1.0022 \) and \( \sigma_1 \cong 0.026 \), which corresponds to the annual parameters of mean return of 1.12% (i.e., mean rate of return of 12%) and a standard deviation of 0.2111 characterizing the U.S. common stock index (Ibbotson 1998).

Table 1 reveals that for various values of one-period correlation, the multiperiod correlation \( |\rho_1| \) indeed decreases as \( n \) increases, but the changes are very small. For example, if \( \rho_1 = 0.7 \) corresponding to weekly data, it only decreases to 0.6996 with four weeks data (about a month), and to 0.6986 with 13 weeks data (about a quarter). Even with about one-year horizon (50 weeks), the correlation is still very close to 0.7, decreasing only to 0.6943. Thus, based on parameters taken from the U.S. stock market corresponding to the common stock index, we can conclude that one can safely shift from one-period to multiperiod parameters (in the relevant range, say one-week to one-year horizons) with almost no effect on \( \rho_1 \). This robust result does not hold with regard to individual stocks that may have a much higher \( \mu \) and \( \sigma \), or even not to other phenomena not necessarily taken from the stock market.

Figure 1 demonstrates the relationship between \( \rho_1 \) and \( \sigma_1 \) for \( n = 4, n = 13, \) and \( n = 50 \) (where \( \mu_1 \cong 1.0022 \) is constant and \( \rho_1 = 0.7 \)). As we can see, for relatively small values of \( \sigma_1 \) the reduction in \( \rho_1 \) is rather minor for all \( n \). However, for large values of \( \sigma_1 \) (which may characterize very risky individual stocks) we get substantial reduction in \( \rho_1 \), particularly for
large $n$ (annual). For example, a calculation of CheckPoint's (CHKP) weekly standard deviation for the year 2000 reveals that $\sigma_y = 0.1423$. For such standard deviation there is a substantial reduction in $\rho_n$, particularly with an annual horizon.

For some phenomena taken from other fields, say biology, where $\mu_y = 1.15$ and $\sigma_y = 0.6$ corresponding to one period (which could be a day, week, or a year), the reduction in $\rho_n$ may be very large. In such a case (not shown in Figure 1), by increasing $n$ from 1 to 4 or 13, we find that $\rho_n$ decreases from 0.7 for $n = 1$ to 0.574 for $n = 4$ and to 0.2816 for $n = 13$. Thus, generally, $|\rho_n|$ may decrease very substantially.

3. The Relationship Between the One-Period and the $n$-Period Linear Regression Slope (Beta)

The slope of the regression is particularly important in many empirical studies. For instance, in the area of finance the slope, called "beta," measures the risk of the asset or the sensitivity of rates of return to changes in some economic factor (see Sharpe 1964, Ross 1976).

Unlike the correlation analysis, in analyzing the impact of the time interval on the regression slope we have to specify which of the variables is additive and which is multiplicative. In this section, we will find the $n$-period regression coefficient, first taking the additive variable and then the multiplicative one as the explanatory variable in the model.

3.1. The Dependent Variable Is Multiplicative

Let $Y = \alpha_y + \beta_y \cdot X$ be a one-period regression model. We construct the following $n$-period regression model:

$$V_n = \alpha_y + \beta_y \cdot W_n,$$

where $V_n$ and $W_n$ are as defined in §2, and $\alpha_y$ and $\beta_y$ are the $n$-period regression coefficients (the regression using $V_n$ and $W_n$).

By definition,

$$\beta_y = \frac{\text{Cov}(W_n, V_n)}{\text{Var}(W_n)} = \frac{\rho_{n} \cdot \sigma_{V_n}}{\sigma_{W_n}},$$

$$= \rho_{n} \cdot \sqrt{\frac{(\mu_x^2 + \mu_y^2)^n - \mu_y^2}{n \cdot \sigma_x^2}}.$$

Substituting $\rho_n$ with the expression in (6), we get

$$\beta_y = \rho_{1} \cdot (\mu_y)^{n-1} \cdot \frac{\sigma_y}{\sigma_x}.$$

Using the relationship $\rho_{1} = \beta_{1} \cdot \sigma_x / \sigma_y$, we obtain

$$\beta_y = \beta_{1} \cdot (\mu_y)^{n-1}. \quad (7)$$

It is interesting to note that $\beta_n$ (as in the correlation case) depends directly on the multiplicative variable $Y$, and not on the additive variable $X$. 

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Theorem 2. Let $\beta_n$ be the slope of the regression as defined above. Then for $\beta_i > 0^2$ we obtain the following results:
1. If $\mu_i > 1$ then $\beta_n$ is monotonically increasing in $n$ and $\lim_{n \to \infty} \beta_n = \infty$.
2. If $|\mu_i| = 1$ then $|\beta_n| = |\beta_i|$ for all $n$ (including the limit).
3. If $0 < \mu_i < 1$ then $\beta_n$ is monotonically decreasing in $n$ and $\lim_{n \to \infty} \beta_n = 0$.
4. If $(-1) < \mu_i < 0$ then $|\beta_n|$ is monotonically decreasing in $n$ and $\lim_{n \to \infty} |\beta_n| = \infty$.
5. If $\mu_i < (-1)$ then $|\beta_n|$ is monotonically increasing in $n$ and $\lim_{n \to \infty} |\beta_n| = \infty$.

These results are obtained directly from Equation (7).

We now turn to the analysis of the $n$-period beta taking the multiplicative variable as the explanatory variable in the model.

3.2. The Explanatory Variable Is Multiplicative
Let $X = \delta + \gamma_i \cdot Y$ be a one-period regression model. Then, using $n$-period data, we run the regression

$$W_n = \delta_n + \gamma_n \cdot V_n,$$

where $V_n$ and $W_n$ are as defined in §2, and $\delta_n$ and $\gamma_n$ are the $n$-period regression coefficients (the regression using $V_n$ and $W_n$).

By definition,

$$\gamma_n = \frac{\text{Cov}(W_n, V_n)}{\text{Var}(V_n)} = \frac{\rho_n \cdot \sigma_{W_n}}{\sigma_{V_n}}.$$

Substituting $\rho_n$ with the expression in (6), we obtain

$$\gamma_n = \rho_1 \cdot (\mu_Y)^{n-1} \cdot \frac{n \cdot \sigma_X \cdot \sigma_Y}{(\sigma_Y^2 + \mu_Y^2)^n}.$$

Using the relationship $\rho_1 = \gamma_i \cdot (\sigma_Y / \sigma_X)$, we obtain

$$\gamma_n = \gamma_i \cdot \frac{n \cdot \sigma_X \cdot \sigma_Y}{(\sigma_Y^2 + \mu_Y^2)^n}.$$

(Theorem 3. Let $\gamma_n$ be the slope of the regression as defined in §3.2. As $n$ approaches infinity, we obtain the following results:
1. If $|\mu_i| \geq 1$ then $\lim_{n \to \infty} \gamma_n = 0$.
2. If $|\mu_i| < 1$ and $\frac{\mu_Y}{(\sigma_Y^2 + \mu_Y^2)^n} \geq 1$,
   then $\lim_{n \to \infty} \gamma_n = +\infty$ if $\gamma_i > 0$.
   $\lim_{n \to \infty} \gamma_n = -\infty$ if $\gamma_i < 0$.
3. If $|\mu_i| < 1$ and $\frac{\mu_Y}{(\sigma_Y^2 + \mu_Y^2)^n} < 1$,
   then $\lim_{n \to \infty} \gamma_n = 0$.

Proof. We will distinguish between two cases corresponding to the mean of $Y$.

Case 1. $|\mu_i| \geq 1$. Using the notations of Theorems 1 and 2, we can compute the $n$-period coefficient $\gamma_n$ as follows:

$$\rho_n^2 = \beta_n \cdot \gamma_n,$$

and hence

$$\gamma_n = \frac{\rho_n^2}{\beta_n}.$$

Because (from Theorems 1 and 2) for $|\mu_i| \geq 1$

$$\lim_{n \to \infty} \rho_n^2 = 0$$

and $\lim_{n \to \infty} \beta_n = \beta_i$,

we get

$$\lim_{n \to \infty} \gamma_n = 0.$$

Note also that as $\rho_n^2$ decreases monotonically with $n$ (see Theorem 1) and $\beta_n$ increases monotonically with $n$, $\gamma_n$ also decreases monotonically with $n$. We would like to emphasize that the monotonicity is intact only to the case $\mu_i \geq 1$, and does not hold for the case $|\mu_i| < 1$, which is discussed next. $\square$

Case 2. $|\mu_i| < 1$. The proof for the case appears in Appendix A.

Numerical Example. Table 2 illustrates the relationship between $\gamma_n$ and $\gamma_i$. In Table 2 we assume that the explanatory variable is multiplicative and that $\gamma_i = 1$ (since $\gamma_n$ is linearly related to $\gamma_i$, this assumption is not restricting). In Table 2 we distinguish between various categories: Columns (1)–(3) correspond to the stock market. In this segment of the table we employ weekly, quarterly, and annual parameters corresponding to the U.S. common stock index. Obviously, in that case $|\mu_i|/(\sigma_Y^2 + \mu_Y^2) < 1$. Column (4) includes results corresponding to a very large $\mu$ and $\sigma$. 1

1 In most economics and finance analysis, the slope $\beta_i$ is positive. However, the analysis is intact for other values of $\beta_i$. If $\beta_i = 0$, then $\beta_n = 0$; if $\beta_i < 0$, then for Case 1 above $\beta_n = -\infty$ and the rest of the results are unchanged.

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Table 2  Regression Coefficient When the Explanatory Variable is Multiplicative—As a Function of the Number of Periods

<table>
<thead>
<tr>
<th>Column</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
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</thead>
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<tr>
<td>STO(Y)</td>
<td>0.0260</td>
<td>0.0963</td>
<td>0.2110</td>
<td>0.6000</td>
<td>0.1000</td>
<td>0.0100</td>
</tr>
<tr>
<td>E(Y)</td>
<td>1.0022</td>
<td>1.0287</td>
<td>1.1200</td>
<td>1.1500</td>
<td>0.9920</td>
<td>0.9500</td>
</tr>
</tbody>
</table>

Number of Periods

<table>
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<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.9975</td>
<td>0.9679</td>
<td>0.8773</td>
<td>0.7654</td>
<td>1.0300</td>
<td>1.0526</td>
</tr>
<tr>
<td>3</td>
<td>0.9950</td>
<td>0.9367</td>
<td>0.7696</td>
<td>0.5830</td>
<td>1.0069</td>
<td>1.1079</td>
</tr>
<tr>
<td>4</td>
<td>0.9925</td>
<td>0.9066</td>
<td>0.6750</td>
<td>0.4420</td>
<td>1.0089</td>
<td>1.1662</td>
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<td>5</td>
<td>0.9900</td>
<td>0.8775</td>
<td>0.5920</td>
<td>0.3386</td>
<td>1.0119</td>
<td>1.2275</td>
</tr>
<tr>
<td>6</td>
<td>0.9875</td>
<td>0.8492</td>
<td>0.5191</td>
<td>0.2506</td>
<td>1.0148</td>
<td>1.2920</td>
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<tr>
<td>7</td>
<td>0.9850</td>
<td>0.8219</td>
<td>0.4552</td>
<td>0.1875</td>
<td>1.0178</td>
<td>1.3599</td>
</tr>
<tr>
<td>8</td>
<td>0.9825</td>
<td>0.7955</td>
<td>0.3991</td>
<td>0.1396</td>
<td>1.0208</td>
<td>1.4314</td>
</tr>
<tr>
<td>9</td>
<td>0.9801</td>
<td>0.7699</td>
<td>0.3499</td>
<td>0.1038</td>
<td>1.0238</td>
<td>1.5067</td>
</tr>
<tr>
<td>10</td>
<td>0.9776</td>
<td>0.7451</td>
<td>0.3067</td>
<td>0.0766</td>
<td>1.0267</td>
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<td>11</td>
<td>0.9752</td>
<td>0.7211</td>
<td>0.2688</td>
<td>0.0564</td>
<td>1.0297</td>
<td>1.6663</td>
</tr>
<tr>
<td>12</td>
<td>0.9727</td>
<td>0.6979</td>
<td>0.2356</td>
<td>0.0414</td>
<td>1.0327</td>
<td>1.7570</td>
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<tr>
<td>13</td>
<td>0.9703</td>
<td>0.6754</td>
<td>0.2064</td>
<td>0.0302</td>
<td>1.0356</td>
<td>1.8494</td>
</tr>
<tr>
<td>14</td>
<td>0.9678</td>
<td>0.6537</td>
<td>0.1809</td>
<td>0.0220</td>
<td>1.0385</td>
<td>1.9466</td>
</tr>
<tr>
<td>15</td>
<td>0.9654</td>
<td>0.6326</td>
<td>0.1585</td>
<td>0.0160</td>
<td>1.0416</td>
<td>2.0490</td>
</tr>
<tr>
<td>20</td>
<td>0.9533</td>
<td>0.5370</td>
<td>0.0817</td>
<td>0.0031</td>
<td>1.0564</td>
<td>2.6472</td>
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<tr>
<td>25</td>
<td>0.9414</td>
<td>0.4558</td>
<td>0.0420</td>
<td>0.0006</td>
<td>1.0712</td>
<td>3.4202</td>
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<tr>
<td>50</td>
<td>0.8840</td>
<td>0.2003</td>
<td>0.0015</td>
<td>0.0000</td>
<td>1.1448</td>
<td>12.3130</td>
</tr>
<tr>
<td>100</td>
<td>0.7794</td>
<td>0.0382</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.2872</td>
<td>159.5802</td>
</tr>
<tr>
<td>500</td>
<td>0.2837</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.7943</td>
<td>≈∞</td>
</tr>
<tr>
<td>1000</td>
<td>0.0796</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.2613</td>
<td>≈∞</td>
</tr>
<tr>
<td>1500</td>
<td>0.0221</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.6692</td>
<td>≈∞</td>
</tr>
</tbody>
</table>

(1) Correspond to weekly data of Common Stock Index.
(2) Correspond to quarterly data of Common Stock Index.
(3) Correspond to yearly data of Common Stock Index.
(4) Correspond to individual stocks or to other phenomena not necessarily taken from the stock market.
(5) This example illustrates a case in which the regression coefficient changes in a nonmonotonic way.
(6) A case in which the regression coefficient grows to infinity.

(That may fit an individual risky asset or a case taken from other fields of research). Columns (5)–(6) where μ_1 < 1 do not correspond to the stock market but may be relevant to other fields of research.

First note that as claimed in Theorem 3, in all cases where μ_1 > 1 (Columns (1)–(4)) γ_n → 0 as n → ∞. Moreover, as claimed in Theorem 3, Columns (1)–(4) of Table 2 reveal that γ_n decreases monotonically as n increases. Note that unlike the changes in the correlation coefficient ρ_n reported in Table 1, here the changes in γ_n are more substantial. For example, if with weekly data γ_1 = 1, with quarterly data it is only 0.9703, dropping to 0.8840 for annual data (Table 2, Column (1)). Thus, while ρ_n is insensitive to the changes in the horizon, the slope γ_n is somewhat more sensitive. The sensitivity in Column (4), which does not correspond to the stock market index but may correspond to individual and very risky assets or to a case not taken from the capital market, is very dramatic.

Now let us turn to the case where μ_1 ≤ 1 (see Columns (5) and (6)). Here we distinguish between two cases: Column (5) corresponds to the case where |μ_n|/[(σ_2^2 + μ_n^2)] < 1 and Column (6) to the case where |μ_n|/[(σ_2^2 + μ_n^2)] > 1. The results here also conform with the claim of Theorem 3. In both cases of Columns (5)
and (6) we have $0 < \mu_1 < 1$. Column (5) corresponds to Case 3 of Theorem 3. Although we prove that in this case $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$, this does not occur at reasonable $n$. However, when we increase $n$ from 1,000 to 1,500 we note a substantial reduction in $\gamma_n$. Moreover, unlike the case where $\mu_1 > 1$, here $\gamma_n$ does not even decrease monotonically. Column (6) corresponds to Case 2 of Theorem 3. Because we selected $\gamma_1 > 0$ we expect $\gamma_n \rightarrow \infty$ as $n \rightarrow \infty$, and indeed, we see that $\gamma_n$ increases with $n$. Here, with the specific data, we also obtain monotonicity, although it is not necessary. Finally, our formulas also relate to negative $\mu_1$, but for the sake of brevity the numerical illustrations corresponding to this case is not reported here.

4. Concluding Remarks
Regression coefficients and correlation are affected by the subdivision of the available data to time intervals over which the random variables are measured. The effect of the selected time interval, in the cases where both variables (the dependent and explanatory variables) are either additive or multiplicative, is well documented in the literature. However, in many cases, one of the variables is additive (e.g., GDP, volume of trade, etc.) and the other variable is multiplicative (e.g., rate of return on financial assets).

This paper analyzes the additive-multiplicative case, showing that the correlation, as well as the regression coefficient, are affected by the selected time interval. We show that $R_n^2$ decreases monotonically as $n$ increases, approaching zero as $n$ goes to infinity. However, with market data characterizing the U.S. common stock index, shifting from weekly to even quarterly parameters only slightly decreases $R_n^2$. For research not relevant to the stock market (i.e., with larger one-period means and standard deviations), the effect of the horizon on $R_n^2$ is substantial. Unlike the negligible effect on the correlation, the effect on the regression slopes is substantial; hence, it is important in such empirical analyses to study the robustness of the effect of changes in the time horizon.

The results of this paper relate to a simple regression model. An extension to the multiple-regression model, as well as the case in which some of the variables are additive and some multiplicative, and for the significance of the coefficients, is the subject of future research.

Acknowledgments
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Appendix
In this Appendix we introduce the proof of Case 2 in Theorem 3 where $|\mu_1| < 1$. We have as before

$$
\gamma_n = \gamma_1 \cdot \frac{n \cdot \sigma_n^2 \cdot (\mu_1)^{y-1}}{(\sigma_1^2 + \mu_1)^{y-1}}
$$

$$
= \gamma \cdot \sigma_1^2 \cdot B,
$$

where

$$
B = \frac{n \cdot (\mu_1)^{y-1}}{\sigma_1^2 + \mu_1}.
$$

Notice that for $\mu_1 = 0$, $\gamma_1 = 0$.

We will compute $\lim_{n \to \infty} B$. Because $\gamma_1 \cdot \sigma_1^2$ is constant, if the limit of $B$ is zero or infinity it will equal the limit of $\gamma_1$.

For simplicity, we will first analyze $1/B$ and then draw a conclusion regarding $B$. In the rest of this proof, we will only refer to the case in which $\mu_1$ is positive. The proof for negative $\mu_1$ is basically the same, but the absolute value of $\mu_1$ is used.

$$
\frac{1}{B} = (\sigma_1^2 + \mu_1)^{(y-1)} / n \cdot (\mu_1)^{y-1} = \frac{(\sigma_1^2 + \mu_1)^{(y-1)} \cdot (\mu_1)^{y-1}}{\sigma_1^2 + \mu_1} \cdot \frac{1}{n}.
$$

Let us denote

$$
A_1 = (\sigma_1^2 + \mu_1)^{(y-1)} \cdot (\mu_1)^{y-1} \cdot \frac{1}{n},
$$

and

$$
A_2 = \frac{(\mu_1)^{y-1}}{n}.
$$

It is obvious that $\lim_{n \to \infty} A_1 = 0$, and therefore the limit of $B$ depends only on the limit of $A_1$. To find the limit of $A_1$, we will use the "Sequence Ratio Criteria":

a. For a given infinite sequence $s_1, s_2, \ldots$,

if from a certain $n$ and on $\frac{s_{n+1}}{s_n} < C < 1$, then $\lim s_n = 0$.

b. For a given infinite sequence $s_1, s_2, \ldots$ ($s_1 > 0$),

if from a certain $n$ and on $\frac{s_{n+1}}{s_n} > C > 1$, then $\lim s_n = \infty$.

Implementing these criteria to the sequence created by changes of $n$ in $1/A_1(n)$, we obtain

$$
\left| \frac{s_{n+1}}{s_n} \right| = \frac{(\sigma_1^2 + \mu_1)^{(y-1)} \cdot (\mu_1)^{y-1}}{(\sigma_1^2 + \mu_1)^{y-1}} \cdot \frac{n + 1}{n} \cdot \frac{1}{\nu / n} \cdot \frac{1}{\nu / n}
$$

We will check all three possible cases of this expression.
Case 2.1. (which corresponds to Case 2 in Theorem 3).
Let
\[ \frac{\mu_i}{(\sigma_i^2 + \mu_i^2)} > C > 1, \]
then
\[ \frac{n+1}{n} \cdot \frac{\mu_i}{(\sigma_i^2 + \mu_i^2)} > C > 1. \]
According to the "sequence ratio criteria" (b),
\[ \lim_{n \to \infty} \frac{1}{\sigma_i^2} = \infty \Rightarrow \lim_{n \to \infty} A_i = 0. \]
Therefore,
\[ \lim_{n \to \infty} \gamma_i = \lim_{n \to \infty} \frac{\gamma_i \cdot \sigma_i^2}{A_i - A_2} = \begin{cases} +\infty & \text{if } \gamma_i > 0 \\ -\infty & \text{if } \gamma_i < 0. \end{cases} \]

Case 2.2. (which corresponds to Case 3 in Theorem 3).
Let
\[ \frac{\mu_i}{(\sigma_i^2 + \mu_i^2)} < C < 1, \]
then for every \( \mu_i \) and \( \sigma_i \) there exists an \( N \) and \( \varepsilon_i \) (so \( \varepsilon_i < 1 - C \)) such that, for every \( n > N \),
\[ \frac{n+1}{n} \cdot \frac{\mu_i}{(\sigma_i^2 + \mu_i^2)} < C + \varepsilon_i < 1. \]
According to the "sequence ratio criteria" (a),
\[ \lim_{n \to \infty} \frac{1}{\sigma_i^2} = 0 \Rightarrow \lim_{n \to \infty} A_i = \infty. \]
Therefore,
\[ \lim_{n \to \infty} \gamma_i = \lim_{n \to \infty} \frac{\gamma_i \cdot \sigma_i^2}{A_i - A_2} = 0. \]

Case 2.3. (which corresponds to Case 2 in Theorem 3).
Let \( \frac{\mu_i}{(\sigma_i^2 + \mu_i^2)} = 1 \), then
\[ \lim_{n \to \infty} \frac{1}{B} = \lim_{n \to \infty} \left[ \frac{(\sigma_i^2 + \mu_i^2)^{n+1}}{\mu_i^n} \left( (\mu_i)^{n+1} - 1 \right) \right] = 0. \]
Therefore,
\[ \lim_{n \to \infty} \gamma_i = \lim_{n \to \infty} \gamma_i \cdot \sigma_i^2 \cdot B = \begin{cases} +\infty & \text{if } \gamma_i > 0 \\ -\infty & \text{if } \gamma_i < 0. \end{cases} \]

References

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