CHAPTER 6: RISK AVERSION AND CAPITAL ALLOCATION TO RISKY ASSETS

PROBLEM SETS

1.  (e)

2.  (b) A higher borrowing is a consequence of the risk of the borrowers’ default. In perfect markets with no additional cost of default, this increment would equal the value of the borrower’s option to default, and the Sharpe measure, with appropriate treatment of the default option, would be the same. However, in reality there are costs to default so that this part of the increment lowers the Sharpe ratio. Also, notice that answer (c) is not correct because doubling the expected return with a fixed risk-free rate will more than double the risk premium and the Sharpe ratio.

3.  Assuming no change in risk tolerance, that is, an unchanged risk aversion coefficient (A), then higher perceived volatility increases the denominator of the equation for the optimal investment in the risky portfolio (Equation 6.12). The proportion invested in the risky portfolio will therefore decrease.

4.  a.  The expected cash flow is: \((0.5 \times 70,000) + (0.5 \times 200,000) = 135,000\)
   With a risk premium of 8% over the risk-free rate of 6%, the required rate of return is 14%. Therefore, the present value of the portfolio is:
   \[ \frac{135,000}{1.14} = 118,421 \]

   b.  If the portfolio is purchased for $118,421, and provides an expected cash inflow of $135,000, then the expected rate of return \([E(r)]\) is derived as follows:
   \[ 118,421 \times [1 + E(r)] = 135,000 \]
   Therefore, \(E(r) = 14\%\). The portfolio price is set to equate the expected rate of return with the required rate of return.

   c.  If the risk premium over T-bills is now 12%, then the required return is:
   \[ 6\% + 12\% = 18\% \]
   The present value of the portfolio is now:
   \[ $135,000/1.18 = $114,407 \]
Chapter 06 - Risk Aversion and Capital Allocation to Risky Assets

d. For a given expected cash flow, portfolios that command greater risk premia must sell at lower prices. The extra discount from expected value is a penalty for risk.

5. When we specify utility by $U = E(r) - 0.5A\sigma^2$, the utility level for T-bills is: 0.07

The utility level for the risky portfolio is: $U = 0.12 - 0.5A(0.18)^2 = 0.12 - 0.0162A$

In order for the risky portfolio to be preferred to bills, the following inequality must hold:

$$0.12 - 0.0162A > 0.07 \Rightarrow A < 0.05/0.0162 = 3.09$$

A must be less than 3.09 for the risky portfolio to be preferred to bills.

6. Points on the curve are derived by solving for $E(r)$ in the following equation:

$$U = 0.05 = E(r) - 0.5A\sigma^2 = E(r) - 1.5\sigma^2$$

The values of $E(r)$, given the values of $\sigma^2$, are therefore:

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\sigma^2$</th>
<th>$E(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.0000</td>
<td>0.05000</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0025</td>
<td>0.05375</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0100</td>
<td>0.06500</td>
</tr>
<tr>
<td>0.15</td>
<td>0.0225</td>
<td>0.08375</td>
</tr>
<tr>
<td>0.20</td>
<td>0.0400</td>
<td>0.11000</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0625</td>
<td>0.14375</td>
</tr>
</tbody>
</table>

The bold line in the following graph (labeled Q6, for Question 6) depicts the indifference curve.
7. Repeating the analysis in Problem 6, utility is now:

\[ U = E(r) - 0.5A\sigma^2 = E(r) - 2.0\sigma^2 = 0.04 \]

The equal-utility combinations of expected return and standard deviation are presented in the table below. The indifference curve is the upward sloping line in the graph above, labeled Q7 (for Question 7).

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( \sigma^2 )</th>
<th>( E(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.0000</td>
<td>0.0400</td>
</tr>
<tr>
<td>0.05</td>
<td>0.0025</td>
<td>0.0450</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0100</td>
<td>0.0600</td>
</tr>
<tr>
<td>0.15</td>
<td>0.0225</td>
<td>0.0850</td>
</tr>
<tr>
<td>0.20</td>
<td>0.0400</td>
<td>0.1200</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0625</td>
<td>0.1650</td>
</tr>
</tbody>
</table>

The indifference curve in Problem 7 differs from that in Problem 6 in both slope and intercept. When \( A \) increases from 3 to 4, the increased risk aversion results in a greater slope for the indifference curve since more expected return is needed in order to compensate for additional \( \sigma \). The lower level of utility assumed for Problem 7 (0.04 rather than 0.05) shifts the vertical intercept down by 1%.

8. The coefficient of risk aversion for a risk neutral investor is zero. Therefore, the corresponding utility is equal to the portfolio’s expected return. The corresponding indifference curve in the expected return-standard deviation plane is a horizontal line, labeled Q8 in the graph above (see Problem 6).
9. A risk lover, rather than penalizing portfolio utility to account for risk, derives greater utility as variance increases. This amounts to a negative coefficient of risk aversion. The corresponding indifference curve is downward sloping in the graph above (see Problem 6), and is labeled Q9.

10. The portfolio expected return and variance are computed as follows:

<table>
<thead>
<tr>
<th>(1) W_Bills</th>
<th>(2) r_Bills</th>
<th>(3) W_Index</th>
<th>(4) r_Index</th>
<th>r_Portion</th>
<th>σ_Portion</th>
<th>σ_Portion²</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>5%</td>
<td>1.0</td>
<td>13.5%</td>
<td>0.135</td>
<td>20%</td>
<td>0.20</td>
</tr>
<tr>
<td>0.2</td>
<td>5%</td>
<td>0.8</td>
<td>13.5%</td>
<td>11.8%</td>
<td>16%</td>
<td>0.16</td>
</tr>
<tr>
<td>0.4</td>
<td>5%</td>
<td>0.6</td>
<td>13.5%</td>
<td>8.4%</td>
<td>8%</td>
<td>0.08</td>
</tr>
<tr>
<td>0.6</td>
<td>5%</td>
<td>0.4</td>
<td>13.5%</td>
<td>4%</td>
<td>4%</td>
<td>0.04</td>
</tr>
<tr>
<td>0.8</td>
<td>5%</td>
<td>0.2</td>
<td>13.5%</td>
<td>2%</td>
<td>2%</td>
<td>0.02</td>
</tr>
<tr>
<td>1.0</td>
<td>5%</td>
<td>0.0</td>
<td>13.5%</td>
<td>0%</td>
<td>0%</td>
<td>0.00</td>
</tr>
</tbody>
</table>

11. Computing utility from \( U = E(r) - 0.5 \times A \sigma^2 = E(r) - 1.5 \sigma^2 \), we arrive at the values in the column labeled \( U(A = 3) \) in the following table:

<table>
<thead>
<tr>
<th>W_Bills</th>
<th>W_Index</th>
<th>r_Portion</th>
<th>σ_Portion</th>
<th>σ_Portion²</th>
<th>U(A = 3)</th>
<th>U(A = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0</td>
<td>0.135</td>
<td>0.20</td>
<td>0.0400</td>
<td>0.0750</td>
<td>0.0350</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>0.118</td>
<td>0.16</td>
<td>0.0256</td>
<td>0.0796</td>
<td>0.0540</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.101</td>
<td>0.12</td>
<td>0.0144</td>
<td>0.0794</td>
<td>0.0650</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.084</td>
<td>0.08</td>
<td>0.0064</td>
<td>0.0744</td>
<td>0.0680</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2</td>
<td>0.067</td>
<td>0.04</td>
<td>0.0016</td>
<td>0.0646</td>
<td>0.0630</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>0.050</td>
<td>0.00</td>
<td>0.0000</td>
<td>0.0500</td>
<td>0.0500</td>
</tr>
</tbody>
</table>

The column labeled \( U(A = 3) \) implies that investors with \( A = 3 \) prefer a portfolio that is invested 80% in the market index and 20% in T-bills to any of the other portfolios in the table.

12. The column labeled \( U(A = 5) \) in the table above is computed from:

\[
U = E(r) - 0.5A \sigma^2 = E(r) - 2.5 \sigma^2
\]

The more risk averse investors prefer the portfolio that is invested 40% in the market index, rather than the 80% market weight preferred by investors with \( A = 3 \).

13. Expected return = \((0.7 \times 18\%) + (0.3 \times 8\%) = 15\%\)

Standard deviation = \(0.7 \times 28\% = 19.6\%\)
14. Investment proportions:
   - 30.0% in T-bills
   - $0.7 \times 25\% = 17.5\%$ in Stock A
   - $0.7 \times 32\% = 22.4\%$ in Stock B
   - $0.7 \times 43\% = 30.1\%$ in Stock C

15. Your reward-to-volatility ratio: $S = \frac{18 - 8}{28} = 0.3571$

   Client's reward-to-volatility ratio: $S = \frac{15 - 8}{19.6} = 0.3571$

16. 

   ![Diagram](Image)

   **CAL (Slope = 0.3571)**

17. a. $E(r_C) = r_f + y[E(r_p) - r_f] = 8 + y(18 - 8)$

   If the expected return for the portfolio is 16%, then:
   
   $16 = 8 + 10y \Rightarrow y = \frac{16 - 8}{10} = 0.8$

   Therefore, in order to have a portfolio with expected rate of return equal to 16%,
   the client must invest 80% of total funds in the risky portfolio and 20% in T-bills.
b. Client’s investment proportions:

- 20.0% in T-bills
- 0.8 × 25% = 20.0% in Stock A
- 0.8 × 32% = 25.6% in Stock B
- 0.8 × 43% = 34.4% in Stock C

c. \( \sigma_C = 0.8 \times \sigma_P = 0.8 \times 28\% = 22.4\% \)

18. a. \( \sigma_C = y \times 28\% \)
   If your client prefers a standard deviation of at most 18%, then:
   \[ y = \frac{18}{28} = 0.6429 = 64.29\% \]
   invested in the risky portfolio

b. \( E(r_C) = 8 + 10y = 8 + (0.6429 \times 10) = 8 + 6.429 = 14.429\% \)

19. a. \[ y^* = \frac{E(r_p) - r_f}{\sigma_P^2} = \frac{0.18 - 0.08}{3.5 \times 0.28^2} = \frac{0.10}{0.2744} = 0.3644 \]
   Therefore, the client’s optimal proportions are: 36.44% invested in the risky portfolio and 63.56% invested in T-bills.

b. \( E(r_C) = 8 + 10y^* = 8 + (0.3644 \times 10) = 11.644\% \)
   \( \sigma_C = 0.3644 \times 28 = 10.203\% \)

20. a. If the period 1926 - 2005 is assumed to be representative of future expected performance, then we use the following data to compute the fraction allocated to equity:
   \[ A = 4, \quad E(r_M) - r_f = 8.39\%, \quad \sigma_M = 20.54\% \]
   (we use the standard deviation of the risk premium from Table 6.8). Then \( y^* \) is given by:
   \[ y^* = \frac{E(r_M) - r_f}{A \sigma_M^2} = \frac{0.0839}{4 \times 0.2054^2} = 0.4972 \]
   That is, 49.72% of the portfolio should be allocated to equity and 50.28% should be allocated to T-bills.

b. If the period 1986 - 2005 is assumed to be representative of future expected performance, then we use the following data to compute the fraction allocated to equity:
   \[ A = 4, \quad E(r_M) - r_f = 8.60\%, \quad \sigma_M = 16.24\% \]
   and \( y^* \) is given by:
   \[ y^* = \frac{E(r_M) - r_f}{A \sigma_M^2} = \frac{0.0860}{4 \times 0.1624^2} = 0.8152 \]
   Therefore, 81.52% of the complete portfolio should be allocated to equity and 18.48% should be allocated to T-bills.
c. In part (b), the market risk premium is expected to be higher than in part (a) and market risk is lower. Therefore, the reward-to-volatility ratio is expected to be higher in part (b), which explains the greater proportion invested in equity.

21. a. \( E(r_C) = 8\% = 5\% + y(11\% - 5\%) \Rightarrow y = \frac{8 - 5}{11 - 5} = 0.5 \)

b. \( \sigma_C = y\sigma_p = 0.50 \times 15\% = 7.5\% \)

c. The first client is more risk averse, allowing a smaller standard deviation.

22. Data: \( r_f = 5\% \), \( E(r_M) = 13\% \), \( \sigma_M = 25\% \), and \( r_f^B = 9\% \)

The CML and indifference curves are as follows:
23. For \( y \) to be less than 1.0 (so that the investor is a lender), risk aversion (\( A \)) must be large enough such that:
\[
y = \frac{E(r_M) - r_f}{A \sigma_M^2} < 1 \Rightarrow A > \frac{0.13 - 0.05}{0.25^2} = 1.28
\]
For \( y \) to be greater than 1.0 (so that the investor is a borrower), risk aversion must be small enough such that:
\[
y = \frac{E(r_M) - r_f}{A \sigma_M^2} > 1 \Rightarrow A < \frac{0.13 - 0.09}{0.25^2} = 0.64
\]
For values of risk aversion within this range, the client will neither borrow nor lend, but instead will hold a complete portfolio comprised only of the optimal risky portfolio:
\[
y = 1 \text{ for } 0.64 \leq A \leq 1.28
\]

24. a. The graph for Problem 22 has to be redrawn here, with:
\( E(r_P) = 11\% \) and \( \sigma_P = 15\% \)

b. For a lending position:
\[A > \frac{0.11 - 0.05}{0.15^2} = 2.67\]
For a borrowing position:
\[A < \frac{0.11 - 0.09}{0.15^2} = 0.89\]
Therefore, \( y = 1 \) for \( 0.89 \leq A \leq 2.67 \)
25. The maximum feasible fee, denoted $f$, depends on the reward-to-variability ratio. For $y < 1$, the lending rate, 5%, is viewed as the relevant risk-free rate, and we solve for $f$ as follows:

$$\frac{11 - 5 - f}{15} = \frac{13 - 5}{25} \Rightarrow f = 6 - \frac{15 \times 8}{25} = 1.2\%$$

For $y > 1$, the borrowing rate, 9%, is the relevant risk-free rate. Then we notice that, even without a fee, the active fund is inferior to the passive fund because:

$$\frac{11 - 9}{15} = 0.13 < \frac{13 - 9}{25} = 0.16$$

More risk tolerant investors (who are more inclined to borrow) will not be clients of the fund even without a fee. (If you solved for the fee that would make investors who borrow indifferent between the active and passive portfolio, as we did above for lending investors, you would find that $f$ is negative: that is, you would need to pay investors to choose your active fund.) These investors desire higher risk-higher return complete portfolios and thus are in the borrowing range of the relevant CAL. In this range, the reward-to-variability ratio of the index (the passive fund) is better than that of the managed fund.
26. a. Slope of the CML = \( \frac{13 - 8}{25} = 0.20 \)
   
The diagram follows.
   
b. My fund allows an investor to achieve a higher mean for any given standard deviation than would a passive strategy, i.e., a higher expected return for any given level of risk.

![CML and CAL diagram]

27. a. With 70% of his money invested in my fund’s portfolio, the client’s expected return is 15% per year and standard deviation is 19.6% per year. If he shifts that money to the passive portfolio (which has an expected return of 13% and standard deviation of 25%), his overall expected return becomes:
   
   \[ E(r_C) = r_f + 0.7[E(r_M) - r_f] = 8 + [0.7 \times (13 - 8)] = 11.5\% \]
   
The standard deviation of the complete portfolio using the passive portfolio would be:
   
   \[ \sigma_C = 0.7 \times \sigma_M = 0.7 \times 25\% = 17.5\% \]
   
   Therefore, the shift entails a decrease in mean from 14% to 11.5% and a decrease in standard deviation from 19.6% to 17.5%. Since both mean return and standard deviation decrease, it is not yet clear whether the move is beneficial. The disadvantage of the shift is that, if the client is willing to accept a mean return on his total portfolio of 11.5%, he can achieve it with a lower standard deviation using my fund rather than the passive portfolio.
To achieve a target mean of 11.5%, we first write the mean of the complete portfolio as a function of the proportion invested in my fund \( y \):

\[
E(r_C) = 8 + y(18 - 8) = 8 + 10y
\]

Our target is: \( E(r_C) = 11.5\% \). Therefore, the proportion that must be invested in my fund is determined as follows:

\[
11.5 = 8 + 10y \Rightarrow y = \frac{11.5 - 8}{10} = 0.35
\]

The standard deviation of this portfolio would be:

\[
\sigma_C = y \times 28\% = 0.35 \times 28\% = 9.8\%
\]

Thus, by using my portfolio, the same 11.5% expected return can be achieved with a standard deviation of only 9.8% as opposed to the standard deviation of 17.5% using the passive portfolio.

b. The fee would reduce the reward-to-volatility ratio, i.e., the slope of the CAL. The client will be indifferent between my fund and the passive portfolio if the slope of the after-fee CAL and the CML are equal. Let \( f \) denote the fee:

\[
\text{Slope of CAL with fee} = \frac{18 - 8 - f}{28} = \frac{10 - f}{28}
\]

\[
\text{Slope of CML (which requires no fee)} = \frac{13 - 8}{25} = 0.20
\]

Setting these slopes equal we have:

\[
\frac{10 - f}{28} = 0.20 \Rightarrow 10 - f = 28 \times 0.20 = 5.6 \Rightarrow f = 10 - 5.6 = 4.4\% \text{ per year}
\]

28. a. The formula for the optimal proportion to invest in the passive portfolio is:

\[
y^* = \frac{E(r_M) - r_f}{A \sigma_M^2}
\]

Substitute the following: \( E(r_M) = 13\%; r_f = 8\%; \sigma_M = 25\%; A = 3.5\):

\[
y^* = \frac{0.13 - 0.08}{3.5 \times 0.25^2} = 0.2286
\]

b. The answer here is the same as the answer to Problem 27(b). The fee that you can charge a client is the same regardless of the asset allocation mix of the client’s portfolio. You can charge a fee that will equate the reward-to-volatility ratio of your portfolio to that of your competition.
CFA PROBLEMS

1. Utility for each investment = \( E(r) - 0.5 \times 4 \times \sigma^2 \)
   We choose the investment with the highest utility value.

<table>
<thead>
<tr>
<th>Investment</th>
<th>Expected return ( E(r) )</th>
<th>Standard deviation ( \sigma )</th>
<th>Utility ( U )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.12</td>
<td>0.30</td>
<td>-0.0600</td>
</tr>
<tr>
<td>2</td>
<td>0.15</td>
<td>0.50</td>
<td>-0.3500</td>
</tr>
<tr>
<td>3</td>
<td>0.21</td>
<td>0.16</td>
<td>0.1588</td>
</tr>
<tr>
<td>4</td>
<td>0.24</td>
<td>0.21</td>
<td>0.1518</td>
</tr>
</tbody>
</table>

2. When investors are risk neutral, then \( A = 0 \); the investment with the highest utility is Investment 4 because it has the highest expected return.

3. (b)

4. Indifference curve 2

5. Point E

6. \((0.6 \times 50,000) + [0.4 \times (-30,000)] - 5,000 = 13,000\)

7. (b)

8. Expected return for equity fund = T-bill rate + risk premium = 6% + 10% = 16%
   Expected return of client’s overall portfolio = \((0.6 \times 16\%) + (0.4 \times 6\%)\) = 12%
   Standard deviation of client’s overall portfolio = 0.6 \times 14\% = 8.4\%

9. Reward-to-volatility ratio = \(\frac{10}{14} = 0.71\)
CHAPTER 6: APPENDIX

1. By year end, the $50,000 investment will grow to: $50,000 \times 1.06 = $53,000

*Without insurance*, the probability distribution of end-of-year wealth is:

<table>
<thead>
<tr>
<th>Probability</th>
<th>Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>No fire</td>
<td>0.999</td>
</tr>
<tr>
<td>Fire</td>
<td>0.001</td>
</tr>
</tbody>
</table>

For this distribution, expected utility is computed as follows:

\[ E[U(W)] = [0.999 \times \ln(253,000)] + [0.001 \times \ln(53,000)] = 12.439582 \]

The certainty equivalent is:

\[ W_{CE} = e^{12.439582} = $252,604.85 \]

*With fire insurance*, at a cost of \( P \), the investment in the risk-free asset is:

\( $(50,000 – P) \)

Year-end wealth will be certain (since you are fully insured) and equal to:

\[ $(50,000 – P) \times 1.06 + $200,000 \]

Solve for \( P \) in the following equation:

\[ $(50,000 – P) \times 1.06 + $200,000 = $252,604.85 \Rightarrow P = $372.78 \]

This is the most you are willing to pay for insurance. Note that the expected loss is “only” $200, so you are willing to pay a substantial risk premium over the expected value of losses. The primary reason is that the value of the house is a large proportion of your wealth.

2. a. With insurance coverage for one-half the value of the house, the premium is $100, and the investment in the safe asset is $49,900. By year end, the investment of $49,900 will grow to: $49,900 \times 1.06 = $52,894

If there is a fire, your insurance proceeds will be $100,000, and the probability distribution of end-of-year wealth is:

<table>
<thead>
<tr>
<th>Probability</th>
<th>Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>No fire</td>
<td>0.999</td>
</tr>
<tr>
<td>Fire</td>
<td>0.001</td>
</tr>
</tbody>
</table>

For this distribution, expected utility is computed as follows:

\[ E[U(W)] = [0.999 \times \ln(252,894)] + [0.001 \times \ln(152,894)] = 12.440225 \]

The certainty equivalent is:

\[ W_{CE} = e^{12.440225} = $252,766.77 \]
b. With insurance coverage for the full value of the house, costing $200, end-of-year wealth is certain, and equal to:

\[
[(50,000 - 200) \times 1.06] + 200,000 = \$252,788
\]

Since wealth is certain, this is also the certainty equivalent wealth of the fully insured position.

c. With insurance coverage for 1½ times the value of the house, the premium is $300, and the insurance pays off $300,000 in the event of a fire. The investment in the safe asset is $49,700. By year end, the investment of $49,700 will grow to: $49,700 \times 1.06 = $52,682

The probability distribution of end-of-year wealth is:

<table>
<thead>
<tr>
<th>Probability</th>
<th>Wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>No fire</td>
<td>0.999</td>
</tr>
<tr>
<td>Fire</td>
<td>0.001</td>
</tr>
</tbody>
</table>

| No fire     | $252,682  |
| Fire        | $352,682  |

For this distribution, expected utility is computed as follows:

\[
E[U(W)] = [0.999 \times \ln(252,682)] + [0.001 \times \ln(352,682)] = 12.4402205
\]

The certainty equivalent is:

\[
W_{CE} = e^{12.44022} = \$252,766.27
\]

Therefore, full insurance dominates both over- and under-insurance. Over-insuring creates a gamble (you actually gain when the house burns down). Risk is minimized when you insure exactly the value of the house.