NOTES FOR FI 4000-VALUATION

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Introduction

The point of this course is to develop tools for analyzing most any basic valuation problem that you might encounter in finance. This is an analytical course but the necessary tools are ones you should have already developed-high school algebra (e.g., how to determine the slope of a straight line and basic statistics (calculating averages, variances and covariances).

What do we mean by valuation? This is a subjective notion of what someone thinks some object (e.g., a security) is worth (in terms of money or some other common denominator), at a point in time. The market price is what the object is actually selling for in the market. A market consists of a group of potential traders. An equilibrium is said to occur when market price = subjective valuation for all potential traders. Otherwise, there is an incentive to trade.

What are we valuing in FI 4000? Mostly financial claims, or securities, of some type, although the tools developed here can be used to value other assets as well (e.g., artwork). Almost all financial claims involve either the exchange of
1. something today (e.g., money) for something else today (e.g., goods) or
2. something later (e.g., money) for something else later (e.g., goods) or
3. something today (e.g., money) for something later (e.g., hopefully more money). The key point here is that, regardless of when the transaction takes place, the terms of trade are typically set today, including contingency provisions for one or the other of the two parties.

What will we cover in this class? The tools developed here will allow you to analyze (determine the value of) how many units of "Y" that one unit of "X" is "worth".

Examples of Unconditional Contracts
a. X=good today; Y=goods tomorrow. Exchange rate = real rate of interest
b. X=money today; Y=money tomorrow. Exchange rate = nominal rate of interest.
c. X=good today; Y=money today. Exchange rate = price level. The inflation rate is the percentage change in this exchange rate over time.
d. X=money #1 today; Y=money #2 today. Exchange rate = spot foreign exchange rate.
e. X=money #1 tomorrow; Y=money #2 tomorrow. Exchange rate = forward foreign exchange rate.
f. X=money tomorrow; Y=goods/securities tomorrow. Exchange rate = forward/futures price.

Examples of Conditional Contracts:
 a. money today for the right to whatever money is left over tomorrow after delivering on other promises. Equity Contract
b. money today for the right to purchase security tomorrow (at an exchange rate fixed today). Call option.
c. money today for the right to sell security tomorrow (at an exchange rate fixed today). 

Put option.

Since few groups can deliver unconditionally, you are almost always faced with risk or uncertainty when you attempt to value securities. The class will mainly deal with risk and not uncertainty, where risk is defined as the chance of an unfavorable (to you) event occurring. An example of risk involves knowing that there are 50 red and 50 black balls in an urn, betting on red but knowing that there is a 50% chance that you will draw black. Uncertainty involves the situation where, not knowing the percentage of red and black balls in the urn, you bet on red but don't know the odds that you will draw black. In the first case you know the distribution of possible outcomes but in the second case you do not know the distribution.

Since valuation is a subjective concept (what is very valuable to me may not be very valuable to you), the tools developed here for purposes of valuation start with an investor profile. The ingredients to a profile include

a. Attitude toward certain money today for certain money tomorrow (subjective rate of interest-time value of money)

b. Attitude toward "randomness" (risk) over a fixed period of time. Investors who ignore randomness are called risk neutral, while those who demand higher compensation for increased randomness are called risk averse.

c. Period of time that is relevant for purposes of decision making (investor horizon- could be infinite)

d. The current endowment (wealth) of the investor.
Valuation involves combining this investor profile with the properties of the financial contract. These properties include
a. the timing of the payoffs, including the last payoff (maturity)
b. the anticipated size of the payoffs at a point in time.
c. the degree of randomness of the payoffs (the risk of the contract).
d. the contingencies, if any, in the contract.

When combined, these eight ingredients can be used to establish the subjective value of a security. The basic trading rule to be used in this class amounts to establishing whether subjective value > market price or subjective value < market price. In the first case you should buy more of the security (go long) while in the second case you should sell the security (go short). Since the subjective value of something depends in part on what else you own, many of the valuation exercises will be done in the context of the what something is worth in a portfolio of securities. In particular, since some of the randomness in payoffs is due to things specific to an individual security (e.g., CEO leaves), risk in the portfolio can be reduced by splitting funds up into many different securities. Diversification eliminates these types of security specific risks, leaving only unavoidable risks.

If all investors behave this way, only nondiversifiable risks will be relevant for purposes of valuation. This is the idea behind the asset pricing models that will be presented later in the course. Before that, however, the class will cover an even more basic tool for purposes of valuing securities—the principle of "no free lunch". In the context of this class that concept will be called "no riskless arbitrage" and it simply means that if you put up none of your own
wealth and take on no randomness, then you should not be able to make a profit.
The next class involves portfolio theory and equilibrium, but before that it will prove useful to look at the notion of riskless arbitrage more closely, since it is probably one of the most basic tools used in valuation. In fact, the absence of riskless arbitrage is essential to establishing equilibrium in financial markets.

The idea here is quite straightforward. Intuitively, if two securities (or assets) generate exactly the same cash flows over time, then they must sell for the same price. If they do not then there is the opportunity for a “free lunch”.

Consider the following simple example. Suppose security X and security Y generate exactly the same cash flow, say $1 next year, but price X > price Y, then you can rig up a situation that has the following three characteristics

1. You put up none of your own money (No investment)
2. You face no “randomness” (No risk) and
3. You generate a positive profit, either now or in the future (Non-negative cash flow)

These are the three components that make up a situation whereby there is “Riskless Arbitrage”. Ruling out these kinds of free lunches (the lack of riskless arbitrage) will prove essential to establishing the value of a number of
securities like options and other so-called derivative securities (i.e., ones whose values depend on the value of some other security).

How, in the example discussed above, could you arrange a situation where 1-3 hold if price X > price Y? Well, first you would try to borrow security X from someone and sell it now. This is called shorting a security. Then you would use the proceeds to purchase Y and have some money left over now. Next year you would receive $1 from security Y but you must return security X to its original owner. But since X is paying $1, you have just enough to buy it (its price at maturity is clearly $1) and return it to its owner. So, you have taken no risk, invested none of your own money and generated a profit (Price X – Price Y)! This can’t last long; eventually Price X = Price Y since everyone will be trying to sell X (driving it’s price down) and buy Y (driving it’s price up). This concept of arbitrage (or no arbitrage) will be used repeatedly throughout the course.

**Portfolio Theory**

It has been stressed that investors like higher expected returns and dislike more "risk". It would be nice to have a uniform measure of risk that could be applied to all securities and bundles of securities equally, regardless of the source of risk. What are the candidates in the "horse race" to be a "good" measure of risk?
First, the most common definition of risk should reflect the fact that an investor faces risk if they face the possibility of realizing outcomes less than what was expected. Second, a reasonable measure of risk would recognize that not all bad outcomes are the same; an outcome somewhat less than expected should be measured as "less risky" than an outcome that is much less than expected. Third, it would be nice if the ranking of more to less risky were independent of the units of measurement; that is the ranking doesn't change whether measured in, say, dollars or pounds.

Interestingly, one common measure that you learned in basic statistics has almost all of these three characteristics. This measure is the standard deviation, which, as you will recall, is the square root of the variance. First, the rankings of standard deviation does not depend on how the units are measured. Second, with the standard deviation, larger deviations from the mean get "more weight" in the risk measure. The only catch is that the standard deviation also gives weight to outcomes that are higher than the mean and more weight to ever larger deviations above the expected outcome.

This problem is not as bad as it sounds. If for example, you are looking at a "symmetric distribution" these shortcomings will not matter. A risk ranking based only on deviations less than the mean will be the same as one where the ranking is based on all of the deviations from the mean. While skewed distributions do arise, and can be important, the standard deviation will work as a good "first" approximation for measuring what is meant by risk. The issue of skewness will come up again in the section covering derivatives.

Review of Means and Standard Deviations:
Recall from statistics that if there are, say, N different states of nature, or scenarios, that can occur, with probabilities $p(1), p(2), \ldots, p(N)$ and $r$ is a random variable that takes on one of N different values, $r(1), r(2) \ldots r(N)$, the expected value or average is given by

$$E(r) = p(1)r(1) + p(2)r(2) + \ldots + p(N)r(N)$$

$$= \sum s p(s)r(s)$$

where $E[.]$ means expected value or average and the term "$\sum s$" means to sum over all possible scenarios ("s"). Furthermore, the variance of $r$ is given by

$$\text{Var}(r) = p(1)(r(1) - E[r])^2 + p(2)(r(2) - E[r])^2 + \ldots + p(N)(r(N) - E[r])^2$$

$$= \sum s p(s)(r(s) - E[r])^2$$

Another way of writing equation (2) that some people find easier to work with is to write $\text{Var}(r)$ as

$$\text{Var}(r) = [\sum s p(s)(r(s))^2] - E(r)^2$$

The standard deviation of $r$, $SD[r]$, is given by

$$SD(r) = \sqrt{\text{Var}(r)}$$

The $p$'s of course, need not be the same. All that is needed is for something (some state) to occur, so $p(1) + p(2) + \ldots + p(N) = 1$ (probabilities must sum to one). If you are, however, working with historical data, the convention is to give each observation equal weight. So if you have N years of data, each year "s" should get an equal "probability", and since the $p$'s must sum to one, you should give each year a probability of $p(s) = 1/N$ for all scenarios "s" when using formulas (1)-(3).
When using historical data the expected value is sometimes called the arithmetic average, or average for short. Also, sometimes in this case $p(s)=1/(N-1)$ is used in calculating $\text{Var}[r]$ in equation (2) but for reasonably large amounts of data (big $N$) this difference doesn't really matter.

Example 1: Suppose that $r$ is the unknown return on a stock and there are three scenarios (or states of the world); a boom, normal growth and a recession, with the associated probabilities and payoffs for this random variable.

<table>
<thead>
<tr>
<th>State(s)</th>
<th>Probability($p(s)$)</th>
<th>Return on stock($r(s)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>boom</td>
<td>30% (.3)</td>
<td>25% (.25)</td>
</tr>
<tr>
<td>normal</td>
<td>50% (.5)</td>
<td>10% (.1)</td>
</tr>
<tr>
<td>recession</td>
<td>20% (.2)</td>
<td>-20% (-.2)</td>
</tr>
</tbody>
</table>

Then using equation (1), the expected value of $r$ is

$$E(r)= p(1)r(1)+ p(2)r(2)+ p(3)r(3)= .3(.25) + .5(.1) + .2(-.2) = .085$$

Likewise, using equation (2) you can calculate the variance as

$$\text{Var}(r)= p(1)(r(1)-E(r))^2+ p(2)(r(2)-E(r))^2+p(3)(r(3)- E(r))^2$$

$$= .3(.25-.085)^2 + .5(.1-.085)^2 + .2(-.2-.085)^2 = .0245$$

and the standard deviation is just

$$\text{SD}(r) = (.0245)^{1/2} = .1566$$
Equations (1) to (3) are very general; once you identity the states or scenarios, the probabilities and the relevant random variable, you can always use these formulas.

Example 2: Historical Data

Suppose instead of the probabilities in Example 1, what you have observed is that over the past three years you have observed returns of 25%, 10% and -20%. Using equations (1) -(3), you would use \( p(s) = \frac{1}{3} \) for all of the outcomes. So in this case you would get

\[
E(r) = p(1)r(1) + p(2)r(2) + p(3)r(3) = \frac{1}{3}(0.25) + \frac{1}{3}(0.1) + \frac{1}{3}(-0.2) = 0.05
\]

\[
\text{Var}[r] = \frac{1}{3}(0.25 - 0.085)^2 + \frac{1}{3}(0.1 - 0.085)^2 + \frac{1}{3}(-0.2 - 0.085)^2 = 0.0362
\]

and

\[
\text{SD}(r) = (0.0362)^{1/2} = 0.1903
\]

Notice that nothing in the calculations change except the probabilities.

In this class the random variables of interest are typically returns on securities and the notation here reflects to the extent possible the notation in Bodie, Kane and Marcus. The return on the safe asset (with \( r \) the same for all scenarios, \( s, \)) is denoted \( r_f \). Since this is known, it will often be referred to as a constant. Another way to say this is to say that the expected return on the safe asset is

\[
E(r) = p(1)r_f + p(2)r_f + p(3)r_f = (p(1) + p(2) + p(3))r_f = r_f
\]
since \( p(1) + p(2) + p(3) = 1 \). Furthermore, the variance is given by
\[
\text{Var}(r) = p(1)(r_f - E(r))^2 + p(2)(r_f - E(r))^2 + p(3)(r_f - E(r))^2 = 0.
\]
since \( E(r) = r_f \) in this case. Also \( SD(r) = (\text{Var}(r))^{1/2} = 0 \).

The working assumption here is that investors like securities, or combinations of securities, with higher values of \( E(r) \) and lower values of \( SD(r) \) (which is the same as lower values of \( \text{Var}(r) \)). In figure 1 \( E(r) \) is plotted on the "Y" axis and \( SD(r) \) is plotted on the "X" axis. So investors are clearly getting "happier" as you move to the Northwest in this picture. Indeed, you could view the goal of portfolio theory as choosing combinations of securities that provide you with risk/return opportunities that are as far to the Northwest as possible. In fact, later on a number (the "Sharpe" ratio) will be developed for the purpose measuring how "good" a particular set of opportunities is in terms of making investors "happier".

Since investors are not limited to buying (or selling) just one security, some provision must be made for calculating means and standard deviations for groups, or portfolios, of securities. Using the following simple rules from statistics for means and standard deviations will prove to be extremely helpful for this purpose.

**Rule 1:**

If \( A \) and \( B \) are constants and \( r_p \) is a random variable, then
\[
(4) \ E(A + B \ r_p) = A + B \ E(r_p)
\]
and
Rule2:
If A and B are constants and \( r_p \) is a random variable, then

\[
SD(A + Br_p) = |B| SD(r_p)
\]

where "|.|" means the absolute value. As mentioned earlier, these two rules will prove extremely useful for purposes of calculating the mean and standard deviation of returns for portfolios of securities, which are the needed inputs for figuring out which combinations are getting you closer to the northwest in figure 1.

The most obvious example, given the discussion so far, would be to calculate the mean and standard deviation of return for a portfolio that consists of some combination of the riskless asset (think of this as a money market fund if you like), with constant return, \( r_f \), and a risky asset with a random return, \( r_p \), where the subscript "p" means risky portfolio.

In this case the "portfolio" consists of only one risky security but the notation will stay the same later when more risky assets are added to the analysis.

Suppose you place "y"% of your funds in the risky security and "1-y"% of your funds in the riskless asset. These weights must sum to one and they are constants; once chosen they are fixed, known numbers, as opposed to being random like \( r_p \). So the return on this combination of securities, \( r_c \) (where the subscript "c" stands for "combination") is just a weighted average of the returns on the two securities; in this case

\[
r_c = (1-y)r_f + y(r_p)
\]
Equation (6) is just a special case of the more general rule for calculating returns when you have a combination of many securities. The following rule will always work for calculating the returns on combinations of securities.

**Rule 3:**

If an investor has the opportunity to invest in "K" securities and places the amount \( y_i \) in the "ith" security, the return on this combination is given by

\[
(7) \quad r_c = y_1(r_1) + y_2(r_2) + \ldots + y_i(r_i) + \ldots + y_K(r_K),
\]

where, of course, \( y_1 + y_2 + \ldots + y_i + \ldots + y_K = 1 \).

Equation (7) is just a formal way to represent the idea that the return on a combination of securities is just the weighted average of the returns on the individual securities. Notice that the \( r \)'s in equation (7) can be random or constants.

Rule 3 will prove useful throughout the course but for the example of one risky and one riskless asset, \( K=2 \) and you get equation (6) (where the subscripts are omitted). Notice that \( y, 1-y \) and \( r_f \) are all constants, while \( r_p \) is a random variable. So, let

\[
(1-y)r_f = A, \quad y = B \quad \text{and} \quad r_p = X.
\]

Then Rule 1 says that the expected value of \( r_c \) can be calculated as

\[
(8) \quad E(r_c) = E(A + BX) = E[(1-y)r_f + (y)r_p] = (1-y)r_f + (y)E(r_p)
\]

and Rule 2 says that the standard deviation of \( r_c \) can be calculated as

\[
(9) \quad SD(r_c) = SD(A + BX) = SD[(1-y)r_f + (y)r_p] = |y| SD(r_p)
\]
Equations (8) and (9), along with Rule 3, is all that will be needed to calculate the means and standard deviations of any combination of securities, regardless of the number of risky assets. How is this going to work? Well, the trick is to first use Rule 3 to calculate \( r_p \), the return on the risky portfolio and then use equations (8) and (9) to calculate the mean and standard deviation of the return on the combination of the riskless security and the risky portfolio of securities. This topic will be discussed more fully later, but it may prove worthwhile to work through some numbers for the one risky and one riskless security case.

**Example 3:** Suppose we take the data from Table 1 as the return on the risky "portfolio" (of one security), so the last column represents \( r_p \). Then it is possible to plot out, for different values of \( y \), the expected return and standard deviation of this combination of securities. Suppose \( r_f = 5\% = 0.05 \). Then using equations (8) and (9), you can plot out \( E(r_C) \) and \( SD(r_C) \)

<table>
<thead>
<tr>
<th>Investment in risky security (y)</th>
<th>((E(r_C)))</th>
<th>(SD(r_C))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.05</td>
<td>0.0</td>
</tr>
<tr>
<td>0.33</td>
<td>0.0616</td>
<td>0.0517</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0763</td>
<td>0.1175</td>
</tr>
<tr>
<td>1</td>
<td>0.085</td>
<td>0.1566</td>
</tr>
<tr>
<td>1.25</td>
<td>0.0938</td>
<td>0.1958</td>
</tr>
</tbody>
</table>
As example 3 makes clear, by changing \( y \) you can plot out all of the relevant values of expected return and risk for this combination of assets. In fact, what you will find is that the line (as a function of \( y \)) is linear. In fact, focusing for the moment only on cases where \( y > 0 \) (\( y < 0 \) represents short sales of the risky asset) so \(|y| = y\), equations (8) and (9) can be written as

\[
(8') \quad E(r_C) = (1-y)r_f + (y)E(r_P) = r_f + y(E(r_P) - r_f)
\]

and

\[
(9') \quad SD(r_C) = y(SD(r_P))
\]

If, on an "X", "Y" graph, you think of equation (8') as \( Y \) and (9') as \( X \), then you know from high school algebra that the rise over the run (as you change \( y \)) is given by

\[
(10) \quad \frac{\Delta E(r_C)}{\Delta y} = E(r_P) - r_f = \text{"\( \Delta \) in Y"}
\]

and

\[
(11) \quad \frac{\Delta SD(r_C)}{\Delta y} = SD(r_P) = \text{"\( \Delta \) in X "}
\]

So the rise over the run really is a constant, independent of \( y \) and the slope of the line is given by

\[
(12) \quad \frac{\Delta \text{in } Y}{\Delta \text{in } X} = \frac{E(r_P) - r_f}{SD(r_P)}
\]

equation (12) defines the "Sharpe ratio", which will be denoted by "S". So in example 3, the slope of this line is \((.085-.05)/.1566 = .2234\).

Table 2 and Figure 2 represents the opportunity set for risk and expected return when the investor is long the risky asset. In fact, for any \( y \) so that \( 1 > y \)
> 0, the investor is said to be long both the risky and the riskless asset. For y > 1, the investor is said to be long the risky and short the riskless asset.
Finally for y < 0, the investor is said to be short the risky asset and long the riskless asset. Table 3 shows the mirror image of Table 2 except that y < 0. The absolute value sign in equation (9) comes into play here the rise over the run now has a negative sign. The Sharpe Ratio S = -.2234 for y < 0. This is plotted in Figure 3.

<table>
<thead>
<tr>
<th>Investment in risky security(y)</th>
<th>(E(rc))</th>
<th>SD(rc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.05</td>
<td>0.0</td>
</tr>
<tr>
<td>-.33</td>
<td>.0385</td>
<td>.0517</td>
</tr>
<tr>
<td>-.75</td>
<td>.0238</td>
<td>.1175</td>
</tr>
<tr>
<td>-1</td>
<td>.015</td>
<td>.1566</td>
</tr>
<tr>
<td>-1.25</td>
<td>.0063</td>
<td>.1958</td>
</tr>
</tbody>
</table>

Clearly, no one who liked expected return and disliked standard deviation would choose to invest in the "opportunity set" where y < 0. There are two general lessons to be learned from this example. The first is

It never makes sense, when viewed in isolation, to short a risky security whose expected return is greater than the riskless rate; i.e., y < 0 if (E(rc)) > rf makes no sense.

This lesson has particular relevance later when you will learn that, when viewed in the context of holding many securities, this seemingly simple idea
may not be true due to the logic of portfolio theory; it is the standard deviation of your whole position, and the not the standard deviation of any particular asset that matters to your happiness in risk/expected return space.

The second lesson to be learned is the notion of efficient portfolio choices. In this example, $y < 0$ is inefficient from a risk/expected return perspective because there are other choices ($y > 0$) that provide a higher expected return for the same level of standard deviation or risk. This leads to a general definition of efficient choices.

An efficient choice for purposes of security selection is one for which there is no other choice with lower risk and the same expected return or higher expected return and the same risk. The set of all efficient choices is called the efficient set or asset allocation line.

So in this example, any $y > 0$ is an efficient choice and any $y < 0$ is an inefficient choice.

Adding More Risky Assets:

In this section you will see that, absent one additional step, calculating risk/return opportunities with many risky assets is computationally identical to what you have just gone through with one risky and one riskless asset. The intermediate step simply involves another application of Rule 3. The trick that will be used involves solving the efficient set of choices in two steps.

Step 1: Decide what percentage of your funds is to be invested in the riskless asset. This determines $1-y$ (% in riskless) and $y$ (total % in risky).
Step 2: Decide what percentage of \( y \) is to be invested in each of the individual risky assets. Call these percentages or weights \( w_i, i = 1, 2, ... K-1 \) (remember that there are \( K \) assets and one of them is riskless), where

\[ w_1 + w_2 + ... + w_i + ... + w_{K-1} = 1. \]

After these two steps you can calculate the return on the risky portfolio using Rule 3 (equation (7)) except use the weights \( w_i \) instead of \( y_i \) (notice that \( w_i = y_i/y \)). This gives you

\[ (13) \quad r_p = w_1 r_1 + w_2 r_2 + ... + w_i r_i + ... + w_{K-1} r_{K-1}. \]

Finally, just use \( y, 1-y, r_p \) and \( r_f \) to calculate expected returns and standard deviations for different combinations of risky and riskless assets just as in the one asset case. The returns on these combinations are calculated the same way and those equations are repeated below

\[ (8) \quad E(r_c) = (1-y) r_f + (y) E(r_p) \]
\[ (9) \quad SD(r_c) = |y| \text{SD}(r_p) \]

**Example 4:** Table 4 below is just Table 1 with the addition of a second risky asset along with a column for the riskless asset.

<table>
<thead>
<tr>
<th>State</th>
<th>Probability ( p(s) )</th>
<th>Return-Stock1 ( r_1 )</th>
<th>Return-Stock2 ( r_2 )</th>
<th>Return-Riskless ( r_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>boom</td>
<td>.3</td>
<td>.25</td>
<td>.20</td>
<td>.05</td>
</tr>
<tr>
<td>normal</td>
<td>.5</td>
<td>.10</td>
<td>.01</td>
<td>.05</td>
</tr>
</tbody>
</table>

Table 4
Suppose that, regardless of the choice of investment in risky assets, you place 40% of that money in asset 2 and 60% in asset 1. So \( w_1 = 0.6 \) and \( w_2 = 0.4 \). You will soon see the rule for optimally choosing the \( w \)'s, but for now just take these numbers as given. So

\[
r_p = 0.6r_1 + 0.4r_2
\]

So \( r_p = 0.6(0.25) + 0.4(0.2) = 0.23 \) with probability 0.3 (a boom), \( r_p = 0.6(0.10) + 0.4(0.01) = 0.064 \) with probability 0.5 (normal) or \( r_p = 0.6(-0.20) + 0.4(-0.05) = -0.14 \), with probability 0.2 (recession). So now you can calculate the expected return and standard deviation of \( r_p \).

\[
E(r_p) = p(1)r_p(1) + p(2)r_p(2) + p(3)r_p(3) = 0.3(0.23) + 0.5(0.064) + 0.2(-0.14) = 0.073
\]

and

\[
\text{Var}(r_p) = p(1)(r_p(1)-E[r_p])^2 + p(2)(r_p(2)-E[r_p])^2 + p(3)(r_p(3)-E[r_p])^2
\]

\[
= 0.3(0.23-0.073)^2 + 0.5(0.064-0.073)^2 + 0.2(-0.14-0.073)^2 = 0.0165
\]

or \( \text{SD}(r_p) = 0.1286 \). Notice that you can rework the Sharpe Ratio,

\[
S = \frac{(0.073-0.05)}{0.1286} = 0.1788,
\]

which is actually less than the Sharpe Ratio when you have only one asset (\( w_1 = 1 \)). So a 60/40 mix between risky asset one and two is clearly not the best mix. The logic here is that since you can always choose to ignore asset 2, your best mix of assets 1 and 2 must be such that the Sharpe Ratio is no lower than that of holding asset 1 (or asset 2 for that matter) alone. After all, you are trying to get to the northwest, and that is equivalent to increasing the slope of the risk/return line. So in general
Increasing the Sharpe Ratio is Equivalent to improving the investment opportunity set. For a given set of assets, the opportunity set with the largest Sharpe ratio is the efficient set.

Naturally, you might wonder how you can determine which way to move the percentages in order to increase the Sharpe Ratio? Table 5 reproduces the returns on the risky assets and the return on the portfolio given the 60/40 mix.

**Table 5**

60/40 mix between asset 1 and 2

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
<th>Return-Stock1</th>
<th>Return-Stock2</th>
<th>Return-portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>boom</td>
<td>.3</td>
<td>.25</td>
<td>.20</td>
<td>.23</td>
</tr>
<tr>
<td>normal</td>
<td>.5</td>
<td>.10</td>
<td>.01</td>
<td>.064</td>
</tr>
<tr>
<td>recession</td>
<td>.2</td>
<td>-.20</td>
<td>-.05</td>
<td>-.14</td>
</tr>
</tbody>
</table>

In order to answer the question you first need to know how to calculate the covariance between two random variables. The covariance is a measure of association between two random variables; If the covariance is positive, then on average the two random variables move in the same direction, while the opposite is true if the covariance is negative.

You need this concept of covariance because, since you are worried about maximizing the expected excess return/risk ratio (the Sharpe Ratio) for your whole portfolio, the desirability of adding some or more of an asset to your
portfolio should naturally be determined based on how it changes the return/risk ratio for your portfolio.

(14) $\text{Cov}(r_i, r_j) = p(1)(r_i(1)-E[r_i])(r_j(1)-E[r_j])+p(2)(r_i(2)-E[r_i])(r_j(2)-E[r_j])+...+p(N)(r_i(N)-E[r_i])(r_j(N)-E[r_j])$

$$= \sum_s p(s)(r_i(s)-E[r_i])(r_j(s)-E[r_j])$$

Another, way to write the covariance is

(14') $\text{Cov}(r_i, r_j) = [\sum_s p(s)r_i(s)r_j(s)]-E[r_j]E[r_i]$  

Equation (14') is usually how I do the calculation, but if you do it correctly, the answers from (14) and (14') will be exactly the same. Notice that the variance is just a special case of the covariance. Using equation (14'), you can calculate $\text{Cov}(r_i, r_i)$. This is given by

(15) $\text{Cov}(r_i, r_i) = \sum_s p(s)(r_i(s)-E[r_i])(r_i(s)-E[r_i])= \sum_s p(s)(r_i(s)-E[r_i])^2$

$$= \text{Var}(r_i)$$

which is just equation (2) for the variance.

**Example 5:** Given the data in Table 5 it is straightforward to calculate the covariance between the returns on the two risky assets and the covariance between the returns on one of the risky assets and the return on the portfolio. For example, $\text{Cov}(r_1, r_2)$ is given, once you determine that $E(r_2) = .055$, by

$\text{Cov}(r_1, r_2) = [\sum_s p(s)r_1(s)r_2(s)]-E[r_1]E[r_2]$

$$=.3(.25)(.2) + .5(.1)(.01) + .2(-.2)(-.05) - (.085)(.055)$$
For a fixed mix of risky assets, you can also calculate the covariance of returns from an individual security and the portfolio return. This will be important in calculating the decision rule as to whether you should add or take away a certain asset in the portfolio.

For example,

\[
\text{Cov}(r_1, r_p) = \sum_s p(s) r_1(s) r_p(s) - E[r_1] E[r_p]
\]

\[
= .3(.25)(.23) + .5(.1)(.064) + .2(-.2)(-.14) - (.085)(.073)
\]

\[= .0198\]

and

\[
\text{Cov}(r_2, r_p) = \sum_s p(s) r_2(s) r_p(s) - E[r_2] E[r_p]
\]

\[
= .3(.2)(.23) + .5(.01)(.064) + .2(-.05)(-.14) - (.055)(.073)
\]

\[= .0115\]

are the covariance of \(r_1\) and \(r_2\) with \(r_p\), respectively, for the 60/40 weights.

Armed with the ability to calculate means, variances and covariances, you can always use the following portfolio decision rule to determine which way the weights should change to increase the expected return/risk ratio.

Now that you know how to calculate the covariance, we can look at an alternative way to calculate the variance of the return on a risky portfolio. I will stick to the two asset case here but the result can be generalized to the \(N\) asset case as well.
(16) \( \text{Var}(r_p) = w_1 \text{Var}(r_1) + w_2 \text{Var}(r_2) + 2w_1w_2 \text{Cov}(r_1, r_2) \)

\[
= w_1 \text{Var}(r_1) + (1-w_1) \text{Var}(r_2) + 2w_1(1-w_1) \text{Cov}(r_1, r_2)
\]

Eexample:

We can use the information in Table 5 to calculate the variance using equation (16). It should turn out to be .0165; just the same as when we calculated it using our earlier method. It is straightforward to calculate \( \text{Var}(r_1) = .0245 \) and \( \text{Var}(r_2) = .0095 \). We already know that \( E(r_1) = .085 \), \( E(r_2) = .055 \) and \( \text{Cov}(r_1, r_2) = .0128 \). We also know that the weights are \( w_1 = .6 \) and \( w_2 = (1-w_1) = .4 \). Therefore, we can calculate

\[
\text{Var}(r_p) = (.6)^2 .0245 + (.4)^2 .0095 + 2(.6)(.4) .0128 = .0165
\]

Which is exactly the same as the alternative method used earlier. The standard deviation is given by \((.0165)^{1/2} = .1284\).

Of course the standard deviation is just the square root of this number.

**Portfolio Decision Rule**

Let \( w_i \) be the percentage of risky assets held in asset \( i \). Then you should follow the following rules in order to maximize the Sharpe ratio

**Increase** \( w_i \) from its current level if

\[
(16) E(r_i) > r_f + [\text{Cov}(r_i, r_p)/\text{Var}(r_p)][E(r_p) - r_f]
\]

**Decrease** \( w_i \) from its current level if
(17) \( E(r_i) < r_f + \frac{\text{Cov}(r_i, r_p)}{\text{Var}(r_p)}[E(r_p) - r_f] \)

Sometimes the term \( \frac{\text{Cov}(r_i, r_p)}{\text{Var}(r_p)} \) is written in short hand notation as

(18) \( \frac{\text{Cov}(r_i, r_p)}{\text{Var}(r_p)} = \beta_{ip} \)

pronounced "beta", where "i" and "p" refer to asset "i" and portfolio "p". So the decision rules can be written more compactly as

**Increase \( w_i \) from its current level if**

(19) \( E(r_i) > r_f + \beta_{ip}[E(r_p) - r_f] \)

**Decrease \( w_i \) from its current level if**

(20) \( E(r_i) < r_f + \beta_{ip}[E(r_p) - r_f] \)

Notice that the only things in equations (19) and (20) that depend on asset "i" are \( E(r_i) \) and \( \beta_{ip} \). Notice that unlike the case of looking at an asset in isolation, portfolio selection rules shows that you may wish to decrease \( w_i \) from current levels (which is the same thing, for fixed \( y \), as reducing \( y_i \)) or even short asset \( i \) (\( w_i < 0 \)).

**Example 6:**

Using the data that you have you already know that

\[
E(r_1) = .085, \ E(r_1) = .055, \ E(r_p) = .073, \ r_f = .05, \ \text{Var}(r_p) = .0165, \\
\text{Cov}(r_1,r_p) = .0198, \ \text{Cov}(r_2,r_p) = .0115, \ \beta_{1p} = .0198/.0165 = 1.2 \quad \text{and} \quad \beta_{2p} = .0115/.0165 = .70.
\]

Applying the rule to both assets you can see that
E(r₁) = .085 > .05 + 1.2(.073 - .05) = .0776, so you should add more to asset 1.

Likewise, doing the calculations for asset 2, you can see that

E(r₂) = .055 < .05 + .70(.073 - .05) = .0661, so you should subtract from asset 2.

Notice also that the weighted average β's sum to 1.0, ie., .6(1.2) + .4(.70) = 1

This will always be true, no matter how many risky assets you look at.

**Many Assets and Investors: The Security Market Line**

If you think of r_f + β_ip[E(r_p) - r_f] as the amount that an investor requires to cover the time value of money (r_f) and the premium for risk in the context of a given portfolio (β_ip[E(r_p) - r_f]), you might call the sum of these two terms the "required" rate of return, call it required(r_i). So, in the context of your portfolio, you should seek out assets whose expected return exceeds the required return and vice versa; sell off or even short assets whose expected return is less than the required return; again in the context of a given portfolio.

Now walk yourself through the following "thought experiment".

Thought 1: Suppose that the whole universe of risky assets was available for investment, so the relevant portfolio of risky assets is the "market" and the return on the portfolio, r_p is the same as the return on market, call it r_m.

Thought 2: Further suppose that investors all "see the same picture". That is, everyone has identical information so they calculate the same values for means, variances and covariances.
Then it will have to be the case, that for all assets $i$, when using portfolio $m$, the following is true in equilibrium

\[
E(r_i) = \text{required}(r_i) = r_f + \beta_{ip}[E(r_p) - r_f]
\]

when portfolio $p$ is the market, or in other words

\[
E(r_i) = \text{required}(r_i) = r_f + \beta_{im}[E(r_m) - r_f]
\]

Equation (22) is typically called the Capital Asset Pricing Model and the plot of $\text{required}(r_i)$ on the "Y" axis against $\beta_{im}$ on the "X" axis is sometimes called the Security Market Line.

Why, given the thoughts 1 and 2, does it have to be that equation (22) holds for all assets? The easiest way to see why it has to hold is to first suppose that it doesn't and show that this contradicts the assumption that you are in an equilibrium situation. For example, suppose that for some asset $i$, it were the case that $E(r_i) > \text{required}(r_i)$. Since everyone sees the same picture, there will be a rush by everyone to purchase asset $i$, which contradicts the assumption that you are in equilibrium. Put another way, there would be no potential sellers of asset $i$ at given prices. In fact, potential sellers would not sell unless what they expect to lose, $E(r_i)$ is less than or equal to what they expect to gain, represented by $\text{required}(r_i)$. The logic can also be worked through for the case where $E(r_i) < \text{required}(r_i)$. In this case everyone would want to be sellers and there would be no potential buyers. So the only equilibrium is one where $E(r_i) < \text{required}(r_i)$.

Notice that the intercept of the function is at $\beta_{im} = 0$ and $\text{required}(r_i) = r_f$ and the slope of the line is $E(r_m) - r_f$. So an asset that does not covary with the market has a required return equal to the riskfree rate. How can this be?
Well, think about the more extreme case, where $\beta_{im} < 0$. In this case, $\text{required}(r_i) < r_f$. The asset actually has a required return of less than the riskless rate. The logic is that, since this asset is counter cyclical, it has a negative covariance with the market, (has high returns when the market return is low), adding this asset to your portfolio, reduces risk by more than that achieved by simply adding an asset whose returns are independent of the market; in this case either the riskless asset or the asset for which $\beta_{im} = 0$. So the general point is that, if $1 > \beta_{im} > 0$, then $\text{E}(r_m) > \text{required}(r_i) > r_f$ and if $\beta_{im} > 1$, $\text{required}(r_i) > \text{E}(r_m)$. In this case the asset has more than average market risk and the required return will reflect this fact.

Of course, in order to calculate the required return, some estimate of $\beta_{im}$, the market portfolio and the riskless rate is required. Usually, time series data in used for this purpose. Moreover, in this class the return on "large company stocks" is typically going to be used as a proxy for the market. The next example shows how these calculations can be done.

**Example 7:**

Below are the returns on large company stocks, small company stocks and T-Bills over the last five years.

<table>
<thead>
<tr>
<th>Year</th>
<th>large stock returns ($r_m$)</th>
<th>small stock returns ($r_i$)</th>
<th>riskless rate ($r_f$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1994</td>
<td>.0131</td>
<td>.0311</td>
<td>.0390</td>
</tr>
<tr>
<td>1995</td>
<td>.3743</td>
<td>.3446</td>
<td>.0560</td>
</tr>
</tbody>
</table>
So, using this historical data, you can calculate all the information you need for the security market line by doing the following. Recall from statistics that you can always fit a straight line between "Y" and "X" that minimizes the sum of squared errors (the line of best fit).

Using the data here, each year gets a probability of 1/5 = .2 = p(s). Furthermore, let $r_i - r_f = Y$ and $r_m - r_f = X$ (given in Table 7) and fit a line of the form

\[
Y = a + bX + e
\]

where $a$ and $b$ are constants and "e" is the error. The best estimate of $b$, call it $\hat{b}$, is given by the formula

\[
\hat{b} = \frac{\text{Cov}(Y,X)}{\text{Var}(X)}
\]

and the best estimate of $a$, call it $\hat{a}$, is given by

\[
\hat{a} = E(Y) - \hat{b} E(X)
\]

<table>
<thead>
<tr>
<th>Year</th>
<th>p(Year)</th>
<th>X = $r_m - r_f$</th>
<th>Y = $r_i - r_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1994</td>
<td>.2</td>
<td>-.0259</td>
<td>-.0079</td>
</tr>
</tbody>
</table>

Table 7
1995  .2  .3183   .2886
1996  .2  .1786   .1241
1997  .2  .2810   .1752
1998  .2  .2372   -.1217

So using the formulas for means, variances and covariances, you can see that
E(X) = .1978, E(Y) = .0917, Var(X) = .0147, Cov(X,Y) = Cov(Y,X) = .0088. So

\[ b = \frac{\text{Cov}(Y,X)}{\text{Var}(X)} = \frac{.0088}{.0147} = .5973 \]

\[ a = E(Y) - b \times E(X) = .0917 - .5973(.1978) = -.0264 \]

So the beta of the small stock portfolio is about .6 and the intercept of this regression is less than zero. You can use the same logic as before in terms of portfolio selection rules. Notice that if CAPM were true, "a" would be zero since

\[ b = \frac{\text{Cov}(Y,X)}{\text{Var}(X)} \]. In this case, \( a < 0 \), so you should sell small stocks and reallocate your portfolio. If \( a > 0 \), you should add more of the security to your portfolio. But this is just another way of looking at the portfolio selection rule that says add if the expected return exceeds the required return and vice versa.

Review of Time Value of Money
Suppose a security promises a cash flow of $\text{CF}_t$ in period $t$, $t=1,2,\ldots,T$ (where “$T$” is the maturity or last payment; $T=\infty$ is possible, of course). For now, assume that the discount rate that equates the present and future value of the cash flow is the same for all $t$; call it $r$. Then the present value of this security, $P$, is given by

\begin{equation}
P = \frac{\text{CF}_1}{1+r} + \frac{\text{CF}_2}{(1+r)^2} + \ldots + \frac{\text{CF}_{T-1}}{(1+r)^{T-1}} + \frac{\text{CF}_T}{(1+r)^T}
\end{equation}

This is a general formulation. Some common special cases are given below.

**Examples:**

i. $\text{CF}_t = A$ for all $t$, $t=1,2,\ldots,T-1, T$. (annuity)

\begin{equation}
P = \frac{A}{1+r} + \frac{A}{(1+r)^2} + \ldots + \frac{A}{(1+r)^{T-1}} + \frac{A}{(1+r)^T}
\end{equation}

In order to get the present value interest factor for an annuity that you know, multiply both sides of equation (2) by $(1+r)$. This gives

\begin{equation}
P(1+r) = A + \frac{A}{1+r} + \ldots + \frac{A}{(1+r)^{T-2}} + \frac{A}{(1+r)^{T-1}}
\end{equation}

Now subtract (2) from (3). This gives

\begin{equation}
Pr = A - \frac{A}{(1+r)^T}
\end{equation}

Dividing both sides of (4) by $r$ and grouping terms gives

\begin{equation}
P = \frac{A}{r}\left[1 - \frac{1}{(1+r)^T}\right]
\end{equation}

(6) You should all be familiar with equation (5) from *Principles of Finance*. For example, if $A = $1, $r = .1$ and $T = 25$, $P = \frac{($1/.1)[1 - 1/(1.1)^{25}]} = $9.07. Also notice that, if $r>0$, then as $T$ goes to infinity, the term $1/(1+r)^T$ goes to zero. So a security paying $A$ per year forever has a value of

\begin{equation}
P = \frac{A}{r} \quad \text{(consol or perpetuity)}.
\end{equation}
So, if $A$ and $r$ are as above, $P = \frac{1}{0.1} = 10$.

ii. $CF_t = A(1+g)^t$ for $t=1, 2, \ldots, T$; (growing annuity). In this case the price is given by

\[
(7) \quad P = \frac{A}{r^*}[1 - \frac{1}{(1 + r^*)^T}]
\]

Where $r^* = \frac{(r - g)}{(1 + g)}$. You can get equation (7) the same way you got equation (5) but, instead of $(1 + r)$, use $(1 + r^*) = \frac{(1 + r)}{(1 + g)}$ as the discount factor and go through the same steps. Notice that $r^* > 0$ if and only if $r > g$ (since $g$ can’t be less than $-1$). So in this case, as $T$ goes to infinity, the term $\frac{1}{(1 + r^*)^T}$ goes to zero and the price of the growing annuity becomes

\[
(8) \quad P = \frac{A}{r^*} = \frac{A}{r - g}
\]

Which, when you define $A(1 + g)$ as the dividend at date 1, is the constant growth model for stock prices that you learned in Principles of Finance. For example, if $T$ is infinity, $r = 0.1$ and $A = 1$, then a non-growing perpetual security is worth $P = \frac{1}{0.1} = 10$. But if $g = 0.05$, then $P = \frac{1(1.05)}{(.1 - .05)} = 21$. This provides an example of how valuation can be broken down into the present value of current income ($10$) and the present value of growth ($11$).

iii. $CF_t = 0$ for $t = 1, 2, \ldots, T-1$ and

\[
CF_t = F \quad \text{for} \quad t = T \quad \text{(zero coupon bond)}
\]

Where “$F$” is the face value, or promised return of principal. This is the simplest kind of bond and will be fundamental to valuation as you go through
this course. The value of a zero coupon bond is, of course, just a simple present value problem. That is,

\[(9) \ P = \frac{F}{(1 + r)^T}\]

iv. \(CF_t = C\) for \(t = 1, 2, \ldots, T-1\) and

\[CF_t = C + F\] for \(t = T\) (ordinary coupon bond)

Where “C” is the coupon payment and “F” is, as before, the face value of the bond. Usually bonds are sold in $1000 denominations and are issued at par value. As you will see next, this means that \(F\) is usually $1000. If the bond makes coupon payment once a year (these are called Euro bonds; bonds issued in the U.S. typically pay half the annual coupon, twice a year (semi-annual)), then

\[(10) \ P = \frac{C}{1 + r} + \frac{C}{(1 + r)^2} + \ldots + \frac{C}{(1 + r)^{T-1}} + \frac{C}{(1 + r)^T} + \frac{F}{1 + r}\]

Using the fact that the coupon payments are an “T” period annuity, equation (10) can be written (use equation (5), pick \(A = C\) and add equation (9)) as

\[(11) \ P = \left(\frac{C}{r}\right)[1 - \frac{1}{(1+r)^T}] + \frac{F}{(1+r)^T}\]

So the value of an ordinary coupon bond is the sum of a T period annuity offering C per year and zero coupon bond promising F at maturity (date T). For example, if \(C = 90\), \(F = 1000\), \(T = 25\) and \(r = .1\), then

\[P = (\frac{90}{.1})[1 - \frac{1}{(1.1)^{25}}] + \frac{1000}{(1.1)^{25}} = 816.93 + 92.30 = 909.23.\]

Some of the terminology from the bond markets will prove useful later on. These include the coupon rate or coupon yield, defined as \(C/F\). In this
example the coupon rate is 90/1000 = .09. Letting \( r_c = C/F \) be the coupon rate, the price in equation (11) can be written as

\[
(12) \quad P = \left( Fr_c/r \right) \left[ 1 - 1/(1+r)^T \right] + F/(1+r)^T
\]

So, if the coupon rate is equal to the discount rate, \( r_c = r \). Using equation (12) you can see that in this case \( P = F \); the bond sells at par. In this example, if \( r = .09 \), then \( P = $1000 \) (check for yourself). If \( r_c < r \), \( P < F \) and it is said that the bond sells at a discount to the par or face value. Finally, if \( r_c > r \), \( P > F \) and the bond is said to sell at a premium relative to the par value. For example, if \( r = .08 \), \( P = $1,106.75 \). By setting the coupon rate equal to current market rates at the time of issue, firm’s can make sure that initially their bonds sell at par (generally $1000, as noted earlier).

**Some Properties of Bond Prices**

The following “fun facts” concerning bond prices will be helpful throughout the course in the sense that they provide fundamental relationships between value, \( P \), and the factors that determine \( P \) (\( T, r, F \) and \( C \)).

*Property 1*: For fixed \( T, F \) and \( C \), \( P \) moves inversely to \( r \).

This is just a reminder from Principles of Finance that bond prices move inversely to discount (or interest) rates.

*Property 2*: For fixed \( T, F \) and \( C \), a decrease in \( r \) results in a larger change in \( P \) than the change in \( P \) associated with an increase in \( r \) of the same absolute magnitude.

Property 2 simply points to the fact that the present value of a future payment is not linear in \( r \). Figure 1 shows why property 2 is true.
Property 3: For fixed C and F, the percentage changes in P associated with a change in r are larger in absolute value, the larger T.

Property 3 simply points out the fact that, for a given change in interest rates, percentage price changes will be larger, the longer the maturity of the bond. You can make sense of this just by looking at equation (9), the equation for a zero coupon bond. Fixing F, a small change in r can cause a large change in P if T is very large, but the change in P will not be very large if T is small. Check this for yourself.

Property 4: For fixed C and F, the percentage changes in P described in property 3 increase at a diminishing rate as T increases.

Property 4 is essentially a statement that, while price changes are increasing in maturity, the difference in the price changes is getting smaller, the longer the maturity. For example, the difference in the percentage price changes for T =4 vs. T= 3 is smaller than the difference in the percentage price changes for T =3 and T =2.

Property 5: For fixed T and F, the percentage change in P, for a given change in r, is larger, the smaller C. The only exceptions to this rule involve cases where the bond has only one remaining payment (T = 1) or the bond is a perpetuity (maturity is infinite).

Property 5 is a statement that reflects the fact that, from a present value point of view, a bond with a large coupon payment is a “shorter term bond” than one with a low (or zero) coupon. You are simply getting more of your money back sooner. The exceptions deal with the case where a.) there is only one payment, so the difference between principal and interest is meaningless or b.) all of the payments come from coupons (no repayment of principal).
You can get a sense of these 5 facts by example. Suppose you have three coupon bonds, with maturities of one, two and three years, respectively. All of these bonds have coupon rates of .1 and F = 1000. You have another bond promising $1210 (= F) in two years, zero otherwise. Initially let r=.1 for all of the bonds. So all of the bonds initially sell for $1000= P.

Now increase r to .11 but leave everything else the same. Then you can verify that now P < 1000 for all of the bonds (verifies property 1). Moreover, you can check that, for the three coupon bonds, P(1 year) > P(2 year) > P(3 year) (verifies property 3). Furthermore, you can check that P(1 year) - P(2 year) > P(2 year) - P(3 year) (verifies property 4). To verify property 5, you can show that P(2 year coupon bond) > P (2 year zero coupon bond). Finally, you can verify property 2 by reworking the problem by changing r from .1 to .09 and checking that the absolute value of the increase in prices is greater than the absolute value of the decrease in prices when r = .11.

**Factors Influencing the Level and Structure of Interest Rates**

This lecture deals with the factors that influence the **level** and **structure** of nominal (or money based) interest rates. The structure of interest rates refers to the relationship between the **time to maturity** and the promised interest rate on the security. The easiest way to discuss these issues is to first consider the case of certainty and then deal with the complications of uncertainty.

**Certainty: Level of Rates**

Inflation
Recall that \( r_f \) is the money rate of interest. What we want to know is what factors go into the determination of \( r_f \). The most basic relationship is sometimes called the "Fisher Hypothesis", which simply asserts that investors do not suffer from money illusion. Money is not valued for itself but rather what goods and services the money can purchase. The Fisher Hypothesis simply states that the money rate of interest should reflect both the "real" discount rate, call it \( R \), as well as changes in the value of money relative to goods and services, which is reflected in the inflation rate. In particular, if it costs \( I_0 \) to buy some good today and \( I_1 \) to buy the same good next year, the inflation rate, call it \( i \), is just the percentage change in the price level, or \( i = (I_1 - I_0)/I_0 \). For example, if a quart of orange juice is \$1\) today and turns out to cost \$1.25 next year, the inflation rate is \( i = (1.25 - 1)/1 = .25 \), or 25%.

The Fisher hypothesis simply asserts that, under certainty, the return from investing in a bond that pays off in money with no inflation adjustment should equal the return on a bond that is inflation protected or else there will be arbitrage. Consider the following two investments

Option 1: Invest \$1 in a default free bond that pays off a known nominal rate, \( r_f \). At the end of the year you will have principal plus interest, or \( 1 + r_f \) as your payoff in money.

Option 2: Invest the same \$1 in a default free bond that promises a rate of \( R \), plus an adjustment (on principal and interest) for inflation, whatever it turns out to be. These will be referred to as index bonds. At the end of the year you will have \((1+R)(1+i)\) as your payoff in money. This bond is similar to the inflation protection bonds (TIP's) issued by the US Treasury.
The Fisher hypothesis simply says that the payoff on Option 1 and Option 2 must be the same; i.e.,

\[(1) \quad 1+r_f = (1+R)(1+i)\]

For example, if \(R = .025\) and \(i = .02\), then \(1+r_f = (1.025)(1.02) = 1.0455\), or \(r_f = .0455\). Otherwise, there will be arbitrage opportunities. For example, suppose that \(r = .04\). In this case, you could borrow \$1 at .04, and invest the dollar in the index bond. Next period the index bond will pay, assuming inflation turns out to be .02, \$1.0455 in money. Can pay back the dollar you borrowed plus interest, or \$1.04, leaving you with a profit of \$.0055. So you have made a profit using none of your own money and taking no risk. People will continue to borrow at "r", driving the nominal rate up, and simultaneously investing in the index bonds, driving "R" down, until equation (1) holds.

Since equation (1) can be written as \(1+r = 1+R+i+Ri\), the Fisher relationship is often approximated (assuming \(Ri\) is "small") by

\[(2) \quad r_f = R + i\]

So, in order to discuss the factors that influence the level of nominal interest rates, one needs to look at the factors that are believed to influence the real rate of interest, \(R\), and the inflation rate, \(i\). Looking first at inflation, it is generally believed that, at least in the long run, inflation is caused by "too much money chasing too few goods". In other words, if the growth rate in money exceeds the growth rate in real goods and services, the economy will experience higher levels of inflation, higher \(i\), and vice-versa if money grows too slowly relative to output. While other factors can influence the inflation rate in the short-term, e.g., oil price shocks, in the long run these simply cause relative price changes e.g., compared to labor, if the supply of money
remains constant. In this case firms will simply substitute labor for oil in their production processes until the relative price changes get corrected.

Turning to the real rate, $R$, it is thought that at least three factors influence the real rate;

(a) the degree of impatience on the part of investors. A higher level of impatience means that, other things the same, investors who have a strong preference for "I want it now", will demand higher returns (higher $R$) for putting off consumption of goods and services.

(b) the level of wealth in the economy. Investors with higher levels of wealth will be willing to lend, other things the same, at lower rates (lower $R$) than those with lower levels of wealth.

(c) the quality of investment opportunities in the economy. As investment opportunities improve, there is greater demand for borrowing and this tends, at least in the short run, to increased levels of $R$, the real rate.

**Certainty: Term Structure of Interest Rates**

**Spot Rates and Yields to Maturity**

In the time value of money review, "the" discount rate, $r$, was taken as a constant, regardless of when the payment occurred in the future. However, in reality it is seldom the case that discount rates are the same for bonds of all maturities. What factors would cause long-term rates to be higher than short-term rates for bonds issued by the same entity, say the U.S. government? In order to look into this question, first define $r_t$ as that rate which can be
earned from buying a security promising CFₜ at date t and zero otherwise at a present value of Pₜ. In other words, rₜ solves

\[ (3) \ Pₜ = \frac{CFₜ}{(1+rₜ)^t} \]

where Pₜ is the present value of the cash flow and CFₜ is the future value at date t. As in any simple present value problem, the issue can also be viewed as one where you invest Pₜ now for t periods at rₜ. This will yield you CFₜ at date t, or

\[ (4) \ Pₜ \ (1+rₜ)^t = CFₜ \]

The rate rₜ is sometimes called the yield, or spot rate, on a zero coupon or pure discount security. The relationship between the rₜ's and t (maturity) is called the term structure of interest rates. Studying these securities is important for a number of reasons. First, the value of more complex securities (like coupon bonds) can be established by knowing the values of these zero coupon securities. This is the concept of value additivity. Second, rₜ represents the proper discount rate to be used to value cash flows to be received at date t. Using the yield to maturity from a more complex security, like a coupon bond, to discount the cash flow at date t generates biased answers for purposes of proper valuation. This is true because, in general, the yield to maturity on a complex (many payment) security with a maturity of "n" years will not equal the spot rate for a payment to be received in year "n". In fact, as the following example shows, the yield to maturity is a complex average of the spot rates.

The following example may help to clarify this notion. Suppose two zero coupon securities are trading in the market. Each promises $100 at maturity
and zero otherwise. The first security matures in one year while the second matures in two years. Current prices are $95 and $90, respectively. Solving for \( r_1 \), one gets that

\[
\frac{100}{(1+r_1)} = 95 \Rightarrow r_1 = \frac{100}{95} - 1 = 0.0526.
\]

Likewise, you can solve for \( r_2 \), since \[
\frac{100}{(1+r_2)^2} = 90 \Rightarrow r_2 = \left(\frac{100}{90}\right)^{1/2} - 1 = 0.0541.
\]

Next, consider the price of an annuity that offers $100 per year for two years. Well, since the cash flows from this annuity are exactly the same as those you would receive from buying one of each of the zero coupon bonds, the price of the annuity must equal $185 (95 + 90) or there would be the possibility of riskless arbitrage.

Suppose, for example, that the annuity was selling for $180. Then you could borrow one each of the zero coupon bonds, sell them now for $95 and $90, respectively (remember that this constitutes two short sales) and promise to return them at maturity. In both cases the bonds will obviously sell for $100 at maturity. Then use the proceeds to buy the annuity for $180, leaving you with $5 in your pocket now. Next year you receive $100 from the annuity but must return the one year bond that is maturing; whose price at maturity $100. Likewise in the second year you receive $100 from the annuity but must return the two year bond, now worth $100. So the future cash flows are a wash, you have money in your pocket and it came from someone else. This can't persist for very long as people buy the annuity and sell the zero coupon bonds.

Given this discussion, it is therefore always possible to find a rate, call it \( y \), such that the discounted cash flows from the annuity equals $185. Specifically, you can solve the yield to maturity problem just like in Principles.
of Finance, i.e., find a \( y = y_2 \) (\( y_n \) is the notation that will be used to define the yield to maturity on a bond whose last payment is made at date \( n \)), so that 
\[
$185 = \frac{100}{1+y_2} + \frac{100}{(1+y_2)^2}
\]
Solving for \( y_2 \) using your calculator gives \( y_2 = 0.0536 \). Notice that \( r_1 = 0.0526 < y_2 = 0.0536 < r_2 = 0.0541 \), so \( y_2 \) really is kind of an "average" of \( r_1 \) and \( r_2 \). But \( y_2 \) should not be used to discount cash flows from date 2 unless the term structure is flat, in which case \( y = r \). Otherwise, using \( y_2 \) to discount cash flows creates a bias relative to proper value of the discounted cash flow to be received at date 2. In this example, \( r_2 > r_1 \) and the term structure is upward sloping, so using \( y_2 \) to discount date 2 cash flows will overstate the value and vice versa when the term structure in downward sloping. (i.e., \( r_2 < r_1 \) in this example).

**Forward/Break-even Rates**

In the above example the term structure is upward sloping; the \( r_t \)'s are getting larger, the larger \( t \). The term structure could however, be downward sloping or even humped (first increasing and then decreasing or vice versa). Two questions need to be addressed. First, why (like today) is the term structure often not flat? and second, as always, is there money to be made if the \( r \)'s are different from each other?

Under certainty the answers are simple. As for the first question, the term structure will be upward sloping today if it is known that interest rates will increase in the future and downward sloping if it is known that rates will increase in the future. As for the second question, the answer in no, as long as there are no possibilities for arbitrage.
To see how this works, work your way through the following mental exercise. Suppose that initially the term structure is flat but that everyone knows (this is certainty) that rates will be higher next year. What are people going to do? Well, you know from property 1 of bond prices that the prices of bonds of all maturities will decrease as rates go up. However, form property 3, longer term bonds will fall by more in price than will short-term bonds. So rational people will try to dump their long-term bonds quicker than short-term bonds. But this selling pressure will drive the price of long-term bonds down and their interest rates up. Today's term structure will now be upward sloping!

This answers the first question, except that you might ask why rates are going to rise next year. Well, from the discussion of the Fisher hypothesis, it must be known that either i) the inflation rate is going to be higher in the future (say because the central bank prints too much money) and/or ii) real rates are going to increase in the future (say because it is known that investors are going to become more "impatient").

In order to answer the second question, you need not look any further than to say that, if there is to be no arbitrage, the return from holding all bonds, no matter the bonds' maturity or how long it is held, must be the same under certainty. Even under uncertainty, however, you can calculate, given today's rates, what future rates must be in order for the return to be the same for all bonds. The future rates that equalize the returns are usually called forward rates or "break-even" rates of interest.

In order to see how this works, suppose you start with zero coupon bonds, paying $1 at maturity, zero otherwise. The bonds have maturities of t=1,
t=2,...,t=T-1 and t = T, respectively (T is the longest term bond). Using the simple present value formula, it is possible, given the prices of these bonds, to calculate the spot rates; the \( r_t \)'s. Suppose you want to compare the return from holding a 2 year bond for one year to that of holding a one year bond for one year. What must the one year bond rate be next year in order for the return to be the same for the two investments? Well, the one year bond will provide a gross return (one plus the return) of \((1 + r_1)\).

The two year bond is currently selling for \$1/(1 + r_2)^2\) \((r_2\) is the two year rate, but these rates are stated in annual terms). However, when it is sold in one year, the two year bond will turn into a one year bond and the sales price will be \$1/(1 + x)\), where "x" is the one year rate, one year from now. So, the gross return from holding this bond for one year will be \[$100/(1 + x)]/[\$100/(1 + r_2)^2] = (1 + r_2)^2/(1 + x)\). So if the gross returns are to be the same on the two bonds, it must be the case that

\[
(5) \quad (1 + r_2)^2 = (1 + r_1)(1 + x)
\]

If you know \( r_2 \) and \( r_1 \), it is easy enough to solve for \( x \). In the earlier example, \( r_2 = .0541 \) and \( r_1 = .0526 \), so \( x = [(1.0541)^2/(1.0526)] - 1 = .0556 \). So, given today's rates, the one year rate one year from now must be .0556 in order for the two investments to earn the same return. So \( x \) is the break-even rate in this example. In general, \( t^fT-t \) will be used to denote the forward rate for a contract to start in period \( t \) and having maturity \( T-t \) \((T > t)\). In this example, \( x = 1^f_1 \); the one year forward rate, to start one year from now. Notice that, under certainty, we could have equivalently said that we wanted the payoff from investing \$1\) in two year bonds and holding it for two years to be the
same as the payoff from putting $1 into a one year bond and rolling over the proceeds into another one year bond.

Given the existence of a three year bond, it is possible to calculate two more forward rates; the one year forward rate, to start two years from now, $2f_1$, as well as the two year forward rate to start one year from now, $1f_2$. For example, suppose that there is another bond promising $100 in three years, zero otherwise and it is selling for $85. Solving for $r_3$ gives $85 = \frac{100}{1 + r_3^3}$, or $r_3 = 0.0557$.

There are two possibilities that need to be dealt with. The first is that an investment of $1 in the three year bond for three years should have the same payoff as that from investing $1 in a one year bond, and reinvesting the money, at maturity, in a two year bond and holding it to maturity; this two year rate is $1f_2$. So this means that $1f_2$ solves

\begin{equation}
1(1 + r_3^3) = (1.0557)^3 = 1(1 + r_1)(1 + 1f_2)^2 = (1.0526)(1 + 1f_2)^2,
\end{equation}

or $1f_2 = 0.0572$. In other words, given today's rates, two year rates next year must be 0.0572 in order to prevent arbitrage. In a similar way, you can find that future one year rate, two years from now that makes the payoff from investing $1 in the three year bond and holding it to maturity and the payoff investing $1 in a sequence of one year bonds. Given that $1f_1$ is known, you can solve for $2f_1$ by solving

\begin{equation}
1(1 + r_3^3) = (1.0557)^3 = 1(1 + r_1)(1 + 1f_1)(1 + 2f_1) = (1.0526)(1.0556)(1 + 2f_1),
\end{equation}

or $2f_1 = 0.0588$. 

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The general rule for calculating forward rates is that if you have a bond with maturity $T$ and another with maturity $t$, $T > t$, then you can always find the forward or break-even rate on a bond to start at $t$ with maturity $T-t$ by solving

$$ (1 + r_T)^T = (1 + r_t)^t(1 + f_{T-t})^{T-t} $$

for $f_{T-t}$. For this example, for $T=3$ and $t=1$ you have solved for $1f_2$ and for $T=3$ and $t=2$, you have solved for $2f_1$. Finally, for $T=2$ and $t=1$, you solved for $1f_1$, which is where the example got started (solving for "x").

**Uncertainty: Level and Structure of Interest Rates**

Although the future is unknown in this case, it is still possible to calculate forward or other "break-even rates". First consider the level of rates. Given that you know, now, the money interest rate, $r$, and the real interest rate, $R$, then you can always solve, using equation (1) for the inflation rate that makes the two returns the same. So even though you don't know $i$ (which is random), you can solve for some $i$, call it $i^*$, so that the returns on the two investments is the same. The value that solves this is

$$ (8) \quad i^* = [(1 + r_t)/(1 + R)] - 1. $$

So, for example, if $r=.05$ and $R=.025$, then you will have been better of buying the index bond if $i > i^* = [1.05/1.025] - 1 = .0244$, and vice versa if $i < .0244$. So, if investors are unconcerned about risk, they will invest in the index bond if they expect inflation higher than .0244 and invest in the non indexed bond if they think that inflation will exceed .0244. But if investors are unconcerned about risk, both securities should have the same expected return, so in this case $i^*$ had better be the expected inflation rate., and the Fisher Hypothesis can be written as
Using exactly the same reasoning, given today's spot rates (the r's), one can always use equation (7) to calculate the break-even or forward rates, even under uncertainty. Again, assuming that investors are unconcerned about risk, one version of the expectations hypothesis says that these forward rates should equal expected future interest rates. This is sometimes referred to as the unbiased expectations hypothesis since forward rates are "unbiased" forecasts of future interest rates.

The difference between uncertainty and certainty is that, even if investors are unconcerned about risk, the expected return cannot be the same for holding different bonds for any maturity. Therefore, the "general" expectations hypothesis noted by Fabozzi simply cannot be true! The expectations hypothesis that will be used here is what has come to be called the "local" expectations hypothesis. This version says that the expected return from holding a security for one period is the same for all securities. This statement implies, because of property 2 of bond prices (remember that present value is not linear in the discount rate), that forward rates will not equal what investor's expect (as opposed to know, in the case of certainty) interest rates to be in the future. The following example may help in clarifying this last point.

Suppose you consider two discount bonds, each promising $1 at maturity and zero otherwise. The maturities are T=1 and T=2, with current prices of $.9259 and $.8264, respectively. Then the known return from holding the one year bond for one year is (1-.9259)/.9259 = .08. Of course the two year bond will be a one year bond in one year. Suppose there is a 50% chance that one
year bond prices, one year from now, will be .9259 and a 50% chance that
they will be .8591. Then the return from holding the two year bond for one
year, since there is no coupon payment, is either \((.9259 - .8264)/.8264 = .1204\)
or \((.8591 - .8264)/.8264 = .0396\). The expected return is just \(.5(.1204) +
.5(.0396) = .08\), the same as that from holding the one year bond for one
year.

Given these possible bond prices, one year interest rates, one year from now,
can also be determined using the simple present value formula. Either it will
be the case that \(.9259 = 1/(1+r)\) or it will be the case that \(.8591 = 1/(1+r)\). So
\(r\) will be either \(.08\) or \(.1640\) and the expected interest rate is \(.5(.08) +
.5(.1640) = .1220\). However, using equation (5) to calculate the one year
forward rate to start one year from now, you get that \(1f_1 = .1204 < .1220\).
This is a general property of the local expectations hypothesis. Forward rates
will be less than expected future interest rates if the expected return from
holding all securities is the same over one period.

What if, instead of being indifferent to risk, investors wanted to avoid
uncertainty. In the example above, holding the two year bond for one year is
a "fair gamble" in the sense that, on average, the return from holding the two
year bond for one year is the same as the known return from holding the one
year bond for one year. Recall that if an investor dislikes risk or uncertainty
they are risk averse, i.e., they are unwilling to take a fair gamble (relative to a
known amount of money).

If an individual investor is risk averse and has a horizon of one year, then the
two year bond is a fair gamble, but risky. This type of risk is often called
interest rate risk but the term used here will be price risk, to signify that, relative to this horizon of one year, the sales price of all longer term bonds is unknown at this investor's horizon.

Put another way, an investor with a one year horizon can eliminate, or immunize him or her self against, risk by purchasing the one year bond. Likewise, an investor with a two year horizon can immunize him or her self against risk by purchasing a two year zero coupon bond. If default free, the return from holding a two year zero coupon bond for two years is known today, regardless of the path of future interest rates. So for this investor the purchase of the one year bond would involve risk. Specifically, the risk that the proceeds will need to be reinvested at some unknown rate. This type of risk is known as reinvestment rate risk. The combination of price risk and reinvestment rate risk is what will be referred to as interest rate risk.

The general rule of thumb is that an investor with horizon $H$ can eliminate interest rate risk by purchasing a zero coupon bond with maturity $T = H$. For $T < H$, the investor faces reinvestment rate risk and for $T > H$, the investor faces price risk. Assuming that all investors want to avoid risk (won't take fair gambles), it is therefore sensible to believe that they will demand a higher expected return (over their horizon) than the known rate of return on a zero coupon bond with maturity $T = H$.

But markets are just made up of a bunch of investors, so which bonds require the higher expected return will depend on the distribution of investor horizons. Suppose, for example, that all investors had horizons exactly equal to ten years. Then all bonds with maturity less than ten years would need to earn more than the ten year rate (the known return on a zero coupon bond
over ten years) because of reinvestment rate risk. Likewise, bonds with maturities of greater than ten years would require a premium due to price risk. In this case we would predict that the return from holding ten year bonds is going to be the lowest among all bonds.

The liquidity preference theory of the term structure basically asserts that most investors have short horizons (small H's), so that on average bonds with higher maturities must earn higher returns because they carry more price risk relative to the short investor horizon. Suppose that the market is dominated by investors with H=1. Then all bonds with maturity T > 1 must earn a premium. The evidence, at least in this century, suggests that the average return from holding bonds is indeed increasing in maturity. When combined with property 3 of bond prices (long term bonds have more price risk than short-term bonds), this is evidence in favor of the idea that investors have short horizons.

When viewed from this perspective, the liquidity preference theory also implies that forward rates will be upward biased estimates of future spot rates if the time value of money influence from property 2 is not "too large". In order to see this, keep in mind that the liquidity preference theory says that all bonds with maturities T > 1 should earn a higher return than that for T = 1. But the gross return from holding a zero coupon bond (paying $1 at maturity) for one year is 1+ r1 and the return from holding the two year bond for one year is [1/(1 + x)][1/(1 + r2)2] =

(1 + r2)2 /(1 + x), where x is the future one period interest rate. So, as long as x < 1f1 = (1 + r2)2/(1 + r1) -1, the return from holding the two year bond will be greater than that from holding a one year bond for one year and vice
versa if $x > 1f1$. So, roughly speaking, the forward rate must usually exceed the future spot rate if the longer term bond is, on average, to have a higher return over one year than the short term bond.

Duration and Convexity:

Value Additivity says that the value of any security is equal to the value of zero coupon bonds. In general, you can write the price of any default free asset promising $CF_t$ at date $t, t = 1, 2, \ldots, T$ as

\[(10) \quad P = CF_1(P_1) + CF_2(P_2) + \ldots + CF_T(P_T)\]

where $P_t = 1/(1 + r_t)^t$ is the price of a zero coupon bond promising $1$ at maturity and zero otherwise. These will often be called simple securities. Clearly, if $CF_t$ is not zero for all $t < T$, an investor with a horizon of $H$ who picks a bond with $T = H$ cannot immunize by buying a non zero coupon bond. While there is no price risk, the intermediate cash flows need to be reinvested at an unknown rate, so that the investor faces reinvestment rate risk. The problem is only amplified if $T < H$, since now the investor faces reinvestment rate risk for both principal and interest.

The only other possibility is $T > H$. In this case the investor faces both price and reinvestment rate risk, but these work in opposite directions. If rates rise above current discount rates and stay there, the bond will have to be sold at a loss. However, the reinvested intermediate payments earn a higher return than at the old rates. Conversely, should rates fall below current rates, the investor realizes a capital gain at sale but a relatively low return on reinvested cash flows. In fact by picking the bond carefully, an investor can rig up a situation where he/she will earn the same return whether rates go up or down since
what is lost in capital (interest) if rates increase (decrease) is offset by what is gained in interest (capital). Thus, by picking the "right" bond, an investor can immunize themselves even with coupon bonds.

What is the "right" bond? Given some simplifying assumptions, we can answer this question. First, let’s assume for the moment that the term structure is flat so that all discount rates are equal, and equal to the yield to maturity, y. Then, given cash flows ,CF_t, and maturity, T, the investor should find a bond such that the investor's horizon, H, is equal to

(11) \[ D = w_1(1) + w_2(2) + \ldots + w_t(t) + \ldots + w_T(T) = \text{Sum}(w_t(t)) \]

where \( w_t = \frac{CF_t/(1+y)}{P} \) is the present value of date t cash flow divided by the sum of the present values (the price = P). Notice that \( w_1 + w_2 + \ldots + w_T = 1 \), so equation (11) is just the time value weighted average maturity of the security and D is known as the Macaulay duration. It is easy to check that the duration of a zero coupon bond with maturity T is \( D = T \). For all other bonds with maturity T, \( D < T \).

Let’s see how one can immunize (lock in) their return by buying a coupon bond with a duration equal to their horizon.

Example: Suppose we have a bond with a coupon of $40 per year and a yield to maturity of \( y = .05 \). The maturity is 3 years and the face value (F) is equal as usual to $1000. This gives a price, \( P \), of $972.76. In order to calculate the duration we must first calculate the weights. In this case \( w_1 = \frac{40/1.05}{972.76} = .0392 \), \( w_2 = \frac{40/(1.05)^2}{972.76} = .0373 \) and \( w_3 = \frac{40+1000}{(1.05)^3}/972.76 = .9235 \). Notice that \( w_1 + w_2 + w_3 = .0392 + .0373 + .9235 = 1 \). The duration is given by \( D = .0392(1) + .0373(2) + .9235(3) = 2.88 \).
If the investor has a horizon $= H$ of 2.88, then this bond will immunize him or her against interest rate changes. To see how this works, we must define the total return from holding the bond for $H$ periods. The general formula is given by

$$P = \text{(total cash flow from bond)}/(1+ \text{return})^H$$
or

$$\text{return} = \left(\text{(total cash flow from bond)}/P\right) - 1$$

Obviously, since $y$ is the yield to maturity, or promised return, if rates stay at 5%, the return will equal $y = .05$. You can check this for yourself. However, suppose that right after purchasing the bond, interest rates jump to 6%. What will be the total return. Well, we must first calculate the total cash flow from the bond. In this case we have total cash flow $= 40(1.06)^{1.88} + 40(1.06)^{.88} + 1040/(1.06)^{.12} = 1119.49$ and the return is given by $\text{return} = (1119.49/972.26)^{1/2.88} = .0501$, which is (approximately) the promised yield to maturity. You can check for yourself that the return will equal the promised return if rates fall to, say, 4% as well.

Why does this work? Notice that reinvestment rate risk (what you can reinvest cash flows at) and price risk work in opposite directions. If $T > H$, then if rates go up, your get more than promised on your reinvestment. However, when you sell the bond after $H$ periods, you take a loss. The opposite situation occurs if rates decline. By setting $D = H$, these two effects exactly offset each other and the realized return is exactly equal to the promised return. It is in this sense that the investor is immunized against rate changes.
The general rule of thumb for comparing the promised to actual return is the following:

If D > H and rates go up after you purchase the bond then return < y

If D > H and rates go down after you purchase the bond then return > y

If D < H and rates go up after you purchase the bond then return > y

If D < H and rates go down after you purchase the bond then return < y

Just like the fun facts for bonds, we have some fun facts for duration.

1. As noted earlier, the duration of a zero coupon bond is equal to its maturity. In other words, \( w_t = 1 \) or \( t = T \) and zero otherwise.

2. Duration is a decreasing function of the coupon rate on a bond. This makes sense because the higher the coupon rate, the more weight that is given to the earlier payments. Since the weights must add to one, this implies less weight for more distant cash flows, thus a lower duration measure.

3. Duration is a decreasing function of the coupon rate on a bond. This makes sense because as the interest rate increases, the present value of distant cash flows decrease more rapidly than those for earlier cash flows. Thus, the weights shift toward earlier cash flows and duration declines. The opposite logic applies when interest rates decline.

4. The duration of a portfolio is just the weighted average of the durations for the individual securities.
In addition to immunization, duration has other useful purposes. The most important is that, when properly modified, duration gives us (approximately) the price risk for a bond. Specifically, in order to calculate % price changes associated with interest rate changes, we typically use what is called modified duration, which is just given by

\[(15) \quad D^* = D/(1+y)\]

What are we going to use D* for? Well, it turns out that, for very small changes in rates, D* can, as noted earlier, be used to get a good approximation to the interest sensitivity of a fixed income security. It particular, you can show that

\[(17) \quad \Delta P/P \approx -D^*(\Delta y)\]

or in words

*the percentage change in the bonds price is approximately equal to the negative of the modified duration multiplied by the change in interest rates.*

So in our earlier three period bond example, D = 2.88, and y = .05, then D* = 2.88/(1.05) = 2.74. Therefore, if rates increase by, say, 100 basis points (so (\Delta y) = .01), then \(\Delta P/P \approx -2.74(0.01)=-.0274\) or put in words a 100 basis point increase in rates will cause this ten year coupon bond to fall in price by approximately 2.74%. A similar decrease in rates would cause the bond to increase in price by a similar percentage. Notice that while the duration for
our example coupon bond is positive, there is no reason why duration cannot be negative. Consider the following example.

Example: Assume that the yield to maturity (or internal rate of return) on a two period project is 5%. Cash flows in year one are $900 and cash flows in year two are $-500. Thus the projects initial price is
\[\frac{900}{1.05} - \frac{500}{(1.05)^2}\]
\[= \$403.63.\]
What is the duration. Well, \(w_1 = \frac{900}{1.05}/403.63 = 2.1236\) and \(w_2 = \frac{-500/(1.05)^2}{403.63} = -1.1236\). Notice that \(w_1 + w_2 = 1\) and the duration is given by
\[D = 2.1236(1) + -1.1236(2) = -.1236.\]
The modified duration is -.1177. What this means is that as interest rates go up, the value of this project also goes up and vice versa.

While modified duration works OK as an approximation for very small changes in rates (formally it works when \(|\Delta y| \approx 0\), it can do a poor job of approximating the actual % change in bond prices when \(|\Delta y|\) is “big”, like it often is in today’s environment. To see what’s going on let’s go back to our ten year coupon bond example. In this case
\[\text{“new } y\text{” = “old } y\text{” + }\Delta y = .05 + .01 = .06\text{ and so the new P at } y = .06\text{ can be found by plugging into the formula for the price of a coupon bond or}\]
\[\text{new } P = \frac{40}{.06[1-1/(1.06)^3]} + \frac{1000}{(1.06)^3} = 946.54\]
So the *actual* percentage change in price for this 100 basis point increase in rates is given by \((946.54 - 972.76)/ 972.76 = -0.0269\). In other words the duration approximation gives an answer that is 5 basis points too large in absolute value in this example. We can say something more general than this and that is that

\[
\frac{\Delta P}{P} > -D*\Delta y \text{ if } \Delta y < 0
\]

\[
\frac{\Delta P}{P} < -D*\Delta y \text{ if } \Delta y > 0
\]

In words, *the duration approximation over predicts price declines as rates increase and under predicts price increases as rates decline.*

Lucky for us there is an adjustment (an ugly, ugly one at that) that can be used to get closer to the actual price decrease. In particular, we can use

\[
(18) \frac{\Delta P}{P} \approx -D*\Delta y + \text{Convexity}(\Delta y)^2/2
\]

where the convexity measure is given by

\[
(19) \text{Convexity} = \frac{[w_1(1)(2) + w_2(2)(3) + \ldots + w_t(t)(t+1) + \ldots + w_T(T)(T+1)]/(1+y)^2}{\text{Sum}[w_t(t)(t+1)]/(1+y)^2}.
\]

Example: For the three period bond example used earlier we have that

\[
\text{convexity} = [.0392(1)(2) + .0373(2)(3) + .9235(3)(4)]/(1.05)^2 = 10.33. \text{ So, we can calculate, for our 100 basis point increase in rates, that } \frac{\Delta P}{P} \approx
\]
\[-D^*(\Delta y) + \text{Convexity}(\Delta y)^2/2 = -2.74(.01) + (10.33/2)(.01)^2 = -.0274 + .0005 \]
\[= -.0269, \text{ which is equal, up to one basis point, to the actual price decline calculated earlier.} \]

Notice that the addition of convexity improves the approximation of modified duration in both directions. By adding a positive number when rates increase, it makes the duration approximation less negative and by adding a positive number for rate increases, it makes the duration approximation more positive. Given our earlier comments, in both cases the approximation is now closer to the actual price change.

**F/X Markets**

Earlier, the idea that interest rates are just exchange rates, *for the same currency, over time*. That is, for example, if you give up $1(US) you get $1(US)(1+r)$ a year from now, where $r$ is the one year interest rate in the US, say on US Treasury bills. In terms of purchasing a US dollar next period, a current US dollar is worth $1/(1+r) = P$, the price of a discount bond, *Now*,

(1) \[ P = 1/(1+r) = \text{the number of US dollars needed now to purchase one US dollar next period.} \]

If, for example, $r=0.1$, then $P=0.90909$, and it takes 0.90909 US dollars today to obtain $1(US)$ next year.
The study of foreign exchange rates is nothing more than the study of the prices at which, say, US dollars can be exchanged for some other currency. This exchange can take place now. This is called the spot foreign exchange contract. Or, by agreement, the exchange can take place in the future, but at an exchange rate agreed upon now. This is called a forward foreign exchange contract. The quotation system used here is the one used by Shapiro; the so-called American quote system, whereby the exchange rate is quoted as the number of US dollars needed to purchase one unit of the foreign currency. So, use the notation

\[(2) \ e = \text{the number of US dollars needed now to purchase one unit of the foreign currency now.}\]

for the spot exchange rate and likewise let

\[(3) \ f = \text{the number of US dollars needed to purchase one unit of the foreign currency. The exchange will take place at some future time}\]

be the forward exchange rate. The interval of time considered here will, as usual be one year (360 days to be exact for this section) but most real forward contracts for foreign exchange trade at shorter intervals than this (forward contracts are available with maturities of 30, 60, 90, 180 and 360 days).

Using this quote convention, a foreign currency is said to be depreciating against the US dollar in the spot or forward market if \( e \) or \( f \) is declining; that is it takes fewer dollars to buy one unit of the foreign currency. Just the opposite would be true if using the European quote system (# of units of foreign currency needed to purchase $1).
Cross Rates: The concept here is that, if you know how many units of currency "A" you need to purchase one unit of currency "B", and you know how many units of currency "A" are needed to purchase one unit of currency "C", it should be straightforward to calculate the cross rate as 

\[
\frac{\#A/\#B}{\#A/\#C} = \#C/\#B.
\]

If the actual exchange rate for currency C to B is different, there will be the opportunity for riskless arbitrage; in this case called triangular arbitrage.

Example: Suppose it takes 150 Japanese yen to purchase $1 in the spot market. Then \(e_Y = 1/150 = .00667\), is the spot dollar to Yen (Y) exchange rate. Furthermore, suppose it takes $2.5 dollars to obtain one British pound (L). Then \(e_L = 2.5\). The cross rate (# of L to get one Y) is given by

\[
.00667/2.5 = .00267.
\]

This cross rate should equal the actual spot rate for pounds to yen. If not, there is an arbitrage opportunity. Suppose, for example, that the actual L/Y exchange rate is .003. What to do? Well, it seems to take more pounds to buy a yen than it should using the cross rate. So the pound is undervalued relative to the yen. Here are the steps for the arbitrage.

Step 1: Borrow $1 and convert it into yen (this whole process is only going to take a minute, so you can mostly ignore the interest cost). Convert it into yen. You receive 150Y.

Step 2: Convert the yen to pounds at the actual exchange rate, to get

\[
150 *.003 = .45L.
\]

Step 3: Convert the pounds back into dollars to get .45*2.5= $1.125 > $1.
As long as the interest rate in dollars is less than 12.5% per minute, you will make a riskless profit from this transaction.

The key here is to purchase the overvalued currency (in this case yen) with dollars, convert the overvalued into the undervalued currency (in this case pounds) and then convert back into dollars. This ability to engage in riskless arbitrage keeps cross rates very close to actual rates. The only catch here is that the strategy is sequential, so, for example, the yen to pound exchange rate in step 2 may change before you have the opportunity to finish step 1.

Another Arbitrage Relationship: Covered Interest Rate Parity

If interest rates are just the exchange of one currency for itself over time, and foreign exchange is just the exchange of one currency for another, you might think that these "exchange" rates are linked. You would be correct. In fact, if both forward and spot f/x contracts trade, the interest rates across countries will be linked via arbitrage arguments. This notion is called covered interest rate parity.

Before looking into the link between $f$, $e$, $r$ and $r^*$, where $r^*$ is the interest rate in the foreign country, some terminology is in order. If $f < e$, the foreign currency is said to trade a forward discount (contracting now, it will take fewer dollars in one year to obtain a unit of the foreign currency than the number of dollars needed now to purchase the currency), while if $f > e$, the currency is said to trade at a forward premium.

What determines whether the currency trades at a discount or premium in the forward market? Covered interest rate parity makes sure that if there are no arbitrage opportunities, $f > e$ if $r > r^*$, or rates in the US are higher than they
are in the foreign country. \( f < e \) if \( r < r^* \). In fact, covered interest rate parity allows for a more specific statement than even this. If there is to be no arbitrage, it must be the case that

\[
(4) \frac{f}{e} = \frac{1 + r}{1 + r^*}
\]

Let's take a closer look at this relationship and see why it makes sense. Consider the following two investment opportunities, starting with $1.

Option 1: Invest the $1 in US security and receive a payoff in dollars next year. In particular, you would get

\[
\text{certain payoff in \$'s next year} = 1(1+r)
\]

Option 2: Take the $1, convert it into the foreign currency now at the spot rate, invest the foreign currency in the foreign bond, and write a forward contract \text{now} to convert back the proceeds of this investment into dollars next year at the forward rate. The payoff in foreign currency will be

\[
\text{certain payoff in foreign currency next year} = \left(\frac{1}{e}\right)(1+r^*)
\]

If you convert this sum (known today) back at \( f \), you will have

\[
\text{certain payoff in \$'s next year} = \left(1\right)\left(\frac{f}{e}\right)(1+r^*)
\]

Covered Interest Rate Parity says that the certain $ payoff from these two investments must be the same. This is the same thing, of course, as saying that equation \( (4) \) must hold. Working through an example may help to clarify why this is true.

Example: Suppose that one year rates in the US are \( r = .05 \), while one year rates in the UK are \( r^* = .10 \). Then if \( e = 2.5 \), covered interest rate parity says
that the forward rate, $f$, must be $(2.5)(1.05)/(1.1) = 2.386$ in order to prevent arbitrage. Suppose that this was not the case and the actual forward rate was, say, 2.25. Then checking back to options 1 and 2, you can safely conclude that at these actual exchange rates, the certain payoff in $'s next year from option 1 $> certain payoff in $'s next year from option 2. In particular, $1.05 > (2.25/2.5)(1.1) = .99$. Under these circumstances it is "free lunch time". So you should go long option 1 and short option.

[Remember to always go long the undervalued asset and short the overvalued asset. But an undervalued asset, under certainty, is simply one that yields a higher return than the overvalued asset for the same amount of investment]. How can this be accomplished. Here are the steps.

**Actions Now:**

Step 1: Borrow 1L today and agree to repay it in one year at a rate r* =10%.

Step 2: Convert the L to dollars at the spot exchange rate $e =2.5$. So you now have $2.5. Invest this in the US bond market at r= 5%.

Step 3: Write a forward contract to mature in one year that allows you to convert $2.625 back into pounds at $f = 2.25$.

**Payoffs Next Year:**

Step 4: Collect $2.625/2.25 = 1.167L$ from your actions in steps 2 and 3.

Step 5: Pay back your loan (in pounds) from step 1. You owe 1.1L

This leaves you with a profit of .067L, regardless of what happens to exchange rates or interest rates over the coming year.
Clearly, investors are going to want to borrow pounds, buy dollars spot, invest in British bonds and sell dollars forward. As there is more selling pressure on $'s forward, the dollar will naturally depreciate in the forward market until the forward rate (remember that this is $/L, so that the dollar depreciating means an increase in $) increases to $ = 2.5 and equation (4) holds.

Keep in mind the importance of the fact that these bonds are assumed to be riskless in the local currency. Think of these bonds as one year US Treasury bills and, say, bonds issued by the British government. This is an extremely important point because if this was not true, you would not be able to construct options 1 and 2 so that the $ payoffs next year were certain. For example, if the securities were instead equity investments in the US and the UK, the returns would be random and you would not know how many $'s to convert back into pounds in Step 3 above.

Given the discussion so far, it should not be surprising that covered interest rate parity holds almost all the time for all of the currencies for which there exist forward markets for currency exchange. In fact, the standard practice by dealers in these markets (ignoring the bid/ask spread) is to quote the forward rate based on knowledge of the interest rates and the spot exchange rate. That is, given $, r and $*, the quote for $ is set so that equation (4) holds.

What about cases where there are no forward contracts? Is there profit to be made? Uncovered interest rate parity.

Even in situations where there is no explicit forward market for foreign exchange, an "implicit" forward rate can be calculated using equation (4), i.e.,
What uncovered interest rate parity says is that if investors are unconcerned about risk, \( f \) (explicit) should be a forecast of the expected future spot exchange rate, call it \( \text{E}[e_1] \), where \( \text{E}[\cdot] \) means expected and \( e_1 \) is the spot exchange rate one year from now. The statement that

\[
(6) f \text{ (explicit)} = \text{E}[e_1]
\]

is a version of the expectations hypothesis for exchange rates. So in this case, while there is no riskless arbitrage opportunities, there are still trading rules that can be developed, much like the case of bonds. These are risky arbitrage strategies and the general rule is if

\[
f \text{ (explicit)} > \text{E}[e_1],
\]

then your forecast is that the dollar will be stronger at date 1 than that implied by current spot exchange rates and interest rates. Using (5) this means that you think that \( (1+ r)/(1+ r^*) > \text{E}[e_1]/e \) and you want to end up selling dollars in the spot market next period. The analogy to term structure is that dollars play the role of long term bonds and the foreign currency that of short term bonds. In this example, you want to go long $ investments and short the pounds, just like if the forward rate is greater than the expected future spot interest rate, you want to buy the longer term bond and borrow and the short term rate. How would you accomplish this? Well, in the example with pounds earlier,
\( f \) (implicit) = 2.386. Suppose your expectation was that the spot exchange rate next year was going to be 2.25 (\( E[e_1] = 2.25 \)), you would go through a strategy similar to that outlined for the case of covered interest rate parity.

Step 1: Same as above

Step 2: Same as above

Step 3: Skip, there is no forward market

Step 4: Collect 2.625/\( e_1 \) L's from steps 1 and 2

Step 5: Same as above. Pay back your loan (in pounds) from step 1. You owe 1.1L.

Notice that if your forecast is correct on average, \( e_1 \) will turn out to 2.25, and this strategy will end up yielding a profit in L's. However, there is risk here. For example, if \( e_1 \) turns out to be 2.5 the dollar appreciated relative to the implicit forward rate. In this case your cash inflow in step 4 will be 2.625/2.5 = 1.05L's, so you end up losing .05L's.

The opposite strategy is used when

\( f \) (implicit) < \( E[e_1] \)

In this case you want to end up buying dollars in the spot market next period. You should then reverse steps 1 to 5.

**What causes exchange rates to fluctuate over time? The International Fisher Hypothesis.**
Recall from lecture 3 that the Fisher Hypothesis says that, in a given currency, it should be the case that the nominal rate reflects the real rate and expected inflation. Using the same notation, this means that in US dollars, it should be the case that

\[(7) \quad 1 + r_f = (1 + R)(1 + E[i])\]

where \(i\) is the inflation rate over the coming year in the US, \(R\) is the real rate (index bond rate) and as usual \(E[.\] means expectation. But if the Fisher hypothesis holds in dollars there is no reason why it should not hold in pounds, so it should be true that

\[(8) \quad 1 + r^*_f = (1 + R^*)(1 + E[i^*]),\]

where the *’s, as usual mean the foreign currency. The international Fisher Hypothesis asserts that if there is a free flow of financial capital, real rates should be the same across countries. In this case \(R = R^*\) and therefore nominal interest rates across countries reflect only differences in expected inflation. If this is true and the expectations hypothesis for exchange rates holds, it must be the case, using (5) - (8), that

\[(9) \quad E[e_{\text{t+1}}]/e = (1 + r_t)/(1 + r^*_t) = (1 + E[i])/(1 + E[i^*])\]

Equation (9) makes perfect economic sense. What it says is that, relative to the current exchange rate, the dollar is expected to depreciate relative to the foreign currency over the next year if investors expect the inflation rate in the US to be higher than that in the foreign country over the coming year and vice versa. But this is sensible since what is being said is that the dollars purchasing power over goods is expected to decrease relative to that of the
foreign currency and therefore the dollar should fetch fewer units of the foreign currency in the future.

**What determines the level of** \( e \) **? Purchasing Power Parity**

The discussion so far has focused on the factors determining the relationship between today's spot rate and forward (or expected future spot) rates. But what determines the level of \( e \)? Well, intuitively, it should have to do with how many $'s vs., say, pounds it takes to purchase something that you want; for example shoes. Purchasing Power Parity (PPP) says something more specific even than this. PPP says that, adjusted for exchange rates, the cost of the same commodity should be the same everywhere. So if a pair of shoes costs $50 in the US and the same pair of shoes cost 20L’s in the UK, the exchange rate for $ to L’s better be 50/20 = 2.5. More generally, if we take a particular price index, say the CPI (consumer price index) in the US and find a similar measure in the foreign country, call it CPI*, then if PPP holds, it should be the case that

\[
(10) \quad e = \frac{\text{CPI}}{\text{CPI}^*}
\]

Simply put, if goods cost more here than in the UK, it should take more than one dollar to get a pound. However, since the price index can be scaled up or down, PPP is usually stated as saying the level of the exchange rate fluctuates up or down depending on whether the price index in the US increases more than in the UK. In particular, if PPP holds next period, then

\[
(11) \quad e_1 = \frac{\text{CPI}_1}{\text{CPI}_1^*}
\]

where the subscript 1 denotes the exchange rate and price indices at date 1. But the inflation rate is just defined as either \( i = (\text{CPI}_1 - \text{CPI})/\text{CPI} \) or \( i^* = \)
(CPI\*1-CPI*)/CPI*, depending on the country. So, what PPP says is something stronger than the International Fisher Hypothesis (equation (9)). PPP says, dividing (11) by (10), that

\[(12) \frac{e}{1/e} = \frac{[CPI_1/CPI_1^*]/[CPI/CPI^*]} = \frac{(1+ i)}{(1 + i^*)}\]

or that the actual (as opposed to just expected) rate fluctuates up or down over time depending on whether the US inflation rate is higher than that in the foreign country (so the dollar depreciates) and vice versa. The intuition behind PPP is that if there is a free and costless flow of physical goods (as opposed to just financial capital) across countries, the "real" (exchange rate adjusted) price of goods will be the same everywhere.

What is the Evidence Regarding All of These Parity Relationships?

Well, as you might expect, those parity relationships that are based on the absence of riskless arbitrage almost always hold. That is

Fact 1: Cross rates almost always equal actual exchange rates. There is seldom the opportunity for triangular arbitrage.

Fact 2: Between Countries for which there are forward foreign exchange contracts, covered interest rate parity almost always holds. The is seldom an opportunity for riskless arbitrage.

These two facts should not be surprising. If they were not true most of the time, there would ample opportunities for a "free lunch".

Fact 3: Uncovered interest rate parity does not always hold.

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Facts 2 and 3 can be viewed graphically in Exhibit 5-10 of Shapiro. Notice that the covered interest rate differentials are almost always zero (fact 2) but the uncovered differentials are consistently positive or negative across the four currencies listed.

Fact 4: The international Fisher Relationship does not generally hold.

The failure of the international Fisher Hypothesis must be due to things like capital controls or tax differentials across countries. Exhibit 7-8 makes clear that countries with high nominal rates (high r*f’s) also have high real rates (high R*’s). The international Fisher Hypothesis, conversely, would imply the R*’s should be the same, regardless of r*f.

Fact 5: PPP fails in the short run but tends to hold over long periods of time.

Governments can, of course, keep the prices of goods in their country higher or lower than world prices by using things like capital controls, import quotas and subsidies for domestic industry. Exhibit 7-4 shows quite clearly that actual exchange rates can differ from PPP predictions for some periods of time. However, notice that over long periods of time the actual series tends to revert to the theoretical PPP line. The moral of the story is that governments can keep prices "out of line" with world prices for awhile, but eventually market forces work to eliminate price discrepancies worldwide.

Derivatives:

A derivative security, as used in the context of this class, simply means a security whose value depends on that of another security, called the underlying. Now since any security can be valued as a function of its future
payoffs, the value of a derivative security must depend on the underlying because its future payoffs must depend on the payoffs from the underlying.

The main tool used for purposes of valuation in this section will be one you already know; how to value things if there is to be no arbitrage. So the real point in this section is to first discuss why someone might want to purchase or sell derivative securities for purposes of speculation or risk management and then discuss how the securities are actually priced.

Before proceeding, it is worthwhile to first look more specifically at what is meant by a derivative security. Almost all derivative claims, no matter how complex, are basically composed of two sorts of contracts; so-called forward contracts and option contracts. You are already familiar with the notion of forward rates for bonds and foreign exchange and you can think of forward contracts as ones that formally allow you to trade on what you think future bond prices (or interest rates) or foreign exchange rates will be in the future. Options are contracts that give you the right to buy or sell a security over some period of time. In both cases the terms of trade are worked out at the beginning of the contract. The following definitions may prove helpful

**Definition of a Forward Contract/ Terminology:**

A forward contract is one that calls for the transfer of some good between two parties at a future point in time, with the delivery price fixed today.

The individual who **obligates** him or her self to **deliver** the good at a price fixed today is called the **short side** of the contract while the individual who
obligates him or her self to take delivery is said to be on the long side of the contract.

The price for future delivery is called the forward price, while naturally, the price for immediate delivery is called the spot price. The key term here is obligates. Next consider the option

Definition of an Option Contract/ Terminology:
The purchaser of an option contract purchases the right to buy (a call) or sell (a put) some good at (a European option) or before (an American option) a future point in time, with the sales price fixed today. Conversely, the writer of a call (put) option, for a fee (called the option price), obligates him or her self to deliver (take delivery of) the good at a price fixed today, which is called the strike or exercise price.

The key term here is obligates as well. However, notice that in this case the purchaser has only rights, which they will clearly exercise only when it is in their monetary interests. The writer of the option must charge an up front fee since they have only obligations and will be called on to meet their obligations only when it is not in their monetary interests.

Why Do These Contracts Exist?

The are two primary reasons why individuals and firms use derivative contracts. The first is to speculate on price movements. While you could speculate by buying or shorting the underlying, derivatives provide a much more cost efficient means of speculating. They provide leverage, as you will
soon see for yourself. The second reason that derivatives exist is for purposes of hedging, or trying to cheaply reduce risk. Hedging and speculation are just two sides of the same coin so derivatives can be viewed as efficient ways to engage in risk management, defined as any overt strategy that changes the risk/return profile of the cash flows from a given underlying position.

Examples:
Speculation and leverage
Suppose that you are willing to regularly lose small sums of money for the chance of a big payoff. Then you might be a candidate to buy or purchase options. Someone else, willing to collect money but risk potentially large losses would be someone who would be a candidate to write options. For example, if you thought a stock was going to increase greatly in value, then you could buy the security but buying a call option would be a much cheaper way (in terms of current cash outflow) to engage in this activity. This is an example of speculation and the leverage effect discussed here is easy to show

Remember that the call option gives you the right to buy the underlying at the strike or exercise price. Suppose the current price of a stock is $100 and you think the price will increase to $150 in the near future. Ignoring dividends, you could a. buy the stock and hope the price rises. If it does, you will have a return on investment of ($150-$100)/$100 = +50%. However, if the price should fall to, say, $50, your return is (50-100)/100 = -50%.
b. Buy the call option. Just for illustration, suppose you buy an option with an exercise price of $100 for, say, $10 (note: just take this figure as given for now; figuring out the correct price for the option is the point of the section on valuing options). Then should the price go to $150, your option is in the money return is (150-100-10)/10 = +400%! However, should the price fall to 50, your option expires worthless (or out of the money) and your return is (0-10)/10 = -100%. So by buying the option instead of the stock you have magnified both your potential gains and losses but this is precisely what leverage does!

Hedging
Futures contracts (a form of forward contracts) actually got started in this country in the Midwest. Farmers were perpetually concerned about the prices they would get for their crops. Since agricultural prices are very volatile, farmers got into the habit of agreeing to sell part of their anticipated crop at a price fixed today. This "locked in" revenue for at least a portion of the crop. So the farmers are short in this example. Conversely, a miller or other merchant who uses agricultural commodities as an input (and other folks who just might want to speculate on the price movements of agricultural commodities) might be willing to take delivery at this fixed price and is said to be on the long side of the contract. Clearly, the farmer's total risk is going to be lower by locking in a price for part of his or her anticipated crop. So the farmer is said to be hedging in this instance.

Notation and Payoffs:

T= maturity/delivery date of the forward contract or option
F_0 = \text{forward price for delivery at date T (determined today)}
S_0 = \text{spot price today.}
S_T = \text{Spot price at date T (random from today's point of view-determined at date T).}
X = \text{strike or exercise price on the option.}
r_T = \text{spot rate for pure discount loans with maturity = T(determined today)}
C_0 = \text{Price of the call option today}
P_0 = \text{Price of the put option today}

Payoffs at date T:

\text{Net/gross payoff at maturity to short position in forward} = F_0 - S_T.

This makes sense because if you agree to deliver for F_0 and S_T turns out to be less than F_0, you have a gain but if S_T turns out to be greater than F_0 you have a loss.

\text{Net/gross payoff at maturity to long position in forward} = S_T - F_0

Notice that in this case, the net and gross payoffs are the same (except for margin-to be discussed later), which is just another way of saying that the initial investment in a forward contract is zero!

\text{Gross payoff at maturity from buying a call option} = S_T - X \quad \text{if}\ S_T > X
= 0 \quad \text{if}\ S_T < X

and
Net payoff at maturity from buying a call option = \(ST - X - C_0\) if \(ST > X\)

\[= -C_0 \quad \text{if } ST < X\]

Similarly,

Gross payoff at maturity from buying a put option = \(X - ST\) if \(ST < X\)

\[= 0 \quad \text{if } ST > X\]

Net payoff at maturity from buying a put option = \(X - ST - P_0\) if \(ST < X\)

\[= -P_0 \quad \text{if } ST > X\]

The gross payoffs from writing calls and puts is just the flip side of the coin

Gross payoff from writing a call option = \(X - ST\) if \(ST > X\)

\[= 0 \quad \text{if } ST < X\]

Net payoff from writing a call option = \(X - ST + C_0\) if \(ST > X\)

\[= C_0 \quad \text{if } ST < X\]

and finally,

Gross payoff from writing a put option = \(ST - X\) if \(ST < X\)

\[= 0 \quad \text{if } ST > X\]

Net payoff from writing a put option = \(ST - X + P_0\) if \(ST < X\)

\[= P_0 \quad \text{if } ST > X\]
Sometimes the gross payoff at maturity or expiration is called the *intrinsic value* of the option since you can show that its value will never be less than this amount.

**Relationship Between Prices:**

**Case 1:**
No intermediate dividends or coupon payments

**Cost of Carry Model**

An investment banker (whose firm is now out of business!) once said that if you understood the cost of carry model, you understood 90% of what was important in pricing derivative securities. This seems a bit much, but nevertheless, the cost of carry model is basic to understanding how derivatives are priced. All the cost of carry model says is that, if you can costlessly contract in forward markets and spot markets and can borrow and lend at spot rates then it must be the case that

*Cost of Carry: The present value of the forward price for delivery at date $T$ must equal the spot price today. That is, if there is to be no arbitrage*

$$F_0/(1+rT)^T = S_0$$

or $F_0 = S_0 (1+rT)^T$

All the cost of carry model says is that, if it is costless to enter into a forward contract, then the payoff at maturity from entering into a long position on a
forward contract must be the same as the payoff from borrowing enough money, at \( rT \) for \( T \) periods, to purchase the stock.

Example:

Suppose that \( S_0 = $10, T=2 \) and \( r_2 = .10 \). Then \( F_0 = 10(1.1)^2 = $12.10 \) in order to prevent arbitrage. To see why, suppose that \( F_0 = $13 \). So the forward contract is "overvalued". Recall that you always want to short the overvalued and buy the undervalued. In this case you go through the following steps

Step 1: Today you enter the short side of the forward contract. Agree to deliver in 2 years at \( F_0=13 \).
Step 2: Today you borrow $10 at 10% interest per year and purchase the security.
Step 3: In two years you deliver the security that you own, collect $13, pay back $10(1.1)^2 = $12.10, leaving a profit of $.90, regardless of what the stock price turns out to be at date 2.

The reason step 3 works is that you already own the security that needs to be delivered on the forward contract. So you have "locked in" your profit today, regardless of what happens to prices over the next two years.

Notice that holding the underlying and hedging (going short in the forward market) is the same as selling the security now and investing the proceeds at the riskless rate of interest. Likewise, holding the underlying and speculating (in this case also going long a forward contract) is equivalent to a "Texas
Hedge", or borrowing at the riskless rate and doubling your bet on the underlying.

Example:
You are a bond fund and currently hold a two year zero coupon bond, promising $100 at delivery and zero otherwise. You can enter into a contract that calls for delivery of a one year bond in one year (T = 1). Suppose that \( r_1 = .05 \) and \( r_2 = .10 \). We know that \( S_0 = 100/(1.1)^2 = 82.64 \) and the no arbitrage forward price, \( F_0 = 82.64(1.05) = 86.78 \). Furthermore, you think that there is a 50/50 chance that one year bond prices one year from now will be either 95.23 or 86.96. Here are your options

Option 1: do nothing, just hold the bond for one period. Your expected return is \( (95.23(.5) + 86.96(.5) - 82.64)/82.64 = 91.095/82.64 - 1 = .1023 \)
And the standard deviation of your return is \( ((95.23-91.095)^2(.5) + (86.96-91.095)^2(.5))^{1/2}/82.64 = .0500 \)

Option 2: hold the bond and go short the forward contract. At date 1 you will receive 86.78 for certain (the forward price) and your return will be \( (86.78 - 82.64)/82.64 = .05 \), which is the riskless rate in risk/expected return space.

Option 3: you can hold the underlying and go long the forward contract.
Your expected return is going to be \( (91.095 + (91.095 - F_0) - 82.64)/82.64 = (182.19-86.78 - 82.64)/82.64 = 1545 \) your standard deviation of returns will be \( .10 \). Check for yourself that if you just borrowed 82.64 for one year at .05 and invested this to buy another two year bond, your expected return and risk would be the same as that outlined in option 3. So the point here is that derivatives may just allow us to do cheaper what we could have done before in terms of moving up and down the expected return/risk line.
Case 2:
Cost of carry when the underlying pays a coupon or dividend (cash yield)
Let \( d \) = the coupon or dividend yield for the underlying (we called this the current yield when we did bond pricing). To keep things simple, let us suppose that the short-term interest rate is a constant. Then since the forward contract does not receive the cash yield, it's price must be reduced by this amount, or

\[
F_0 = S_0 (1 + r - d)^T
\]

Example: Suppose you have a forward contract to deliver a perpetuity at date 1, so \( T = 1 \). Let the current short rate, \( r = .05 \) and assume that the perpetuity has a rate of .1 right now with a coupon payment of \( C = $100 \) per year. Then \( S_0 = 100/.1 = 1000 \). Furthermore, \( d = C / S_0 = 100/1000 = .1 \). So \( F_0 \) must be \( F_0 = 1000(1.05 -.1)^1 = 950 \).

Interest Rate Swaps
Interest Rate Swaps are nothing more than a series of one period forward contracts, settled in arrears. They were created in part because there are no exchange traded contracts with maturity greater than a year so hedging interest rate exposure for a long period of time created a challenge.
Consider a fixed for floating rate swap agreement, where one party pays the fixed rate \( r_{\text{fixed}} \) and receives an random floating rate, call it \( r_{\text{floating}(t-1)} \), where the \( t-1 \) means that this is the floating rate at \( t-1 \) to be paid at date \( t \), and \( t \) runs, as usual from 1 to \( T \), the maturity of the contract. Let \( $N \) be the "notional principal" on which this contract is written (The # of dollars on which to
calculate the interest payments/receipts. Then the total cash flow to pay fixed, receive floating, is given by

\[
\text{Cash flow to pay fixed, receive floating party} = \$N(r_{\text{floating}(0)} - r_{\text{fixed}}) + \$N(r_{\text{floating}(1)} - r_{\text{fixed}}) + \ldots + \$N(r_{\text{floating}(T-2)} - r_{\text{fixed}}) + \$N(r_{\text{floating}(T-1)} - r_{\text{fixed}}).
\]

Notice that if floating rates turn out to be higher on average than fixed rates, this party makes a positive cash flow while the reverse would be true if floating rates turn out to be less than fixed on average. Since this swap is just a bunch of forwards and forwards require no initial investment, swaps should be priced so that the initial cash flow to both parties is zero. How can this be achieved? Well, there are two problems to tackle.

1. The floating rates, beyond today's rate (date 0) are not known and must be estimated.

2. Given estimates in 1., a fixed rate must be found that makes the present value of the expected payments equal to zero for both parties.

The first problem is solved by using forward rates from today's term structure to estimate the expected floating rates. In particular, we have that

\[
r_{\text{floating}(0)} = r_1 = \text{today's one year rate (known)}
\]

\[
E(r_{\text{floating}(1)}) = 1 f_1 = (1 + r_2)^2/(1 + r_1) - 1
\]

\[
E(r_{\text{floating}(2)}) = 2 f_1 = (1 + r_3)^3/(1 + r_2)^2 - 1
\]

\[
E(r_{\text{floating}(T-1)}) = T-1 f_1 = (1 + r_T)^T/(1 + r_{T-1})^{T-1} - 1
\]
Where E(.) means what participants are using this as their forecast. Keep in mind that if liquidity preference is true, f will overstate actual floating rates on average and those who receive fixed and pay floating will tend to gain.

Given these forecasts it is easy to find the fixed rate that makes the present value (when discounted at today's spot rates) of this contract equal to zero.

In particular, solve the following for \( r_{\text{fixed}} \)

\[
0 = \left(\frac{N(r_1 - r_{\text{fixed}})}{1 + r_1}\right) + \left(\frac{N(t_1 f_1 - r_{\text{fixed}})}{(1 + r_2)^2}\right) + \ldots + \left(\frac{N(T_1 f_1 - r_{\text{fixed}})}{(1 + r_{T-1})^T}\right)
\]

**Example:** Suppose that \( T=4 \) (I will sometimes call this a "three" year swap since only three payments will be made in this example) and the notional amount is \( N = 10,000,000 \). The current term structure is \( r_1 = .05 \), \( r_2 = .075 \) and \( r_3 = .10 \). Then

\[
r_{\text{floating}(0)} = r_1 = .05
\]

\[
E(r_{\text{floating}(1)}) = f_1 = (1 + r_2)^2/(1 + r_1) - 1 = (1.075)^2/(1.05) - 1 = .1006
\]

\[
E(r_{\text{floating}(2)}) = 2f_1 = (1 + r_3)^3/(1 + r_2)^2 - 1 = (1.1)^3/(1.075)^2 - 1 = .1518
\]

So we need to solve the following equation for \( r_{\text{fixed}} \)

\[
0 = (10,000,000(.05 - r_{\text{fixed}}))/(1.05) + (10,000,000(.1006- r_{\text{fixed}}))/(1.075)^2 + (10,000,000(.1518- r_{\text{fixed}}))/(1.1)^3 \]. This gives

\[
0 = .2487 - r_{\text{fixed}} (2.569) \text{ (notice that } N = 10,000,000 \text{ cancels out), so}
\]

\[
r_{\text{fixed}} = .0968 \text{, which we already knew would be less than } .1 \text{, the three year spot rate (recall that "yields" are less than spot rates if the term structure is upward sloping).}
\]

**Put/Call Parity**
Even though figuring out the price of a particular option is covered in the next section, these calculations require some assumptions about the price of the underlying as it evolves over time. The so-called put/call parity relationship, on the other hand, simply involves the lack of arbitrage. The standard version of put/call parity relates the prices of puts and calls on the same security with the same exercise price and the same maturity date.

Definition of Put/Call Parity:

If two options, a put and call, have a common exercise price of X and a common maturity of T, then in the absence of arbitrage it must be the case that

\[ P_0 = C_0 + \frac{X}{(1 + r_T)^T} - S_0 \]

where \( r_T \) is the spot rate for date T pure discount bonds.

To see why this works, consider the following two investment alternatives

alternative 1:
Buy a call and write a put at an exercise price of X and a maturity T.

Initial Cash Flow \( P_0 - C_0 \)

Payoffs at date T

<table>
<thead>
<tr>
<th></th>
<th>( S_T &lt; X )</th>
<th>( S_T &gt; X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>payoff from buying call</td>
<td>0</td>
<td>( S_T - X )</td>
</tr>
</tbody>
</table>


payoff from writing put \( ST - X \) \( 0 \)

total payoff from option 1 \( ST - X \) \( ST - X \)
at maturity

alternative 2:
Borrow \( X/(1+rT)^T \) for \( T \) periods at the rate \( rT \), and invest in one share of the stock.

Initial Cash Flow \( X/(1+rT)^T - S_0 \)

Payoffs at date \( T \)
if \( ST < X \)
if \( ST > X \)

payoff from buying stock \( ST \) \( ST \)

payoff from loan \( -(X/(1+rT)^T)(1+rT)^T \) \( -(X/(1+rT)^T)(1+rT)^T \)

total payoff from option 2 \( ST - X \) \( ST - X \)
at maturity

By now you know that if alternatives 1 and 2 have the same total payoff at maturity, then if there are no other cash flows, the initial cash flows associated with the two alternatives must also be equal or else there are arbitrage possibilities.

Example: Suppose \( S_0 = 10 \), \( T = 2 \), \( X = 12.10 \), \( r_2 = .10 \) and \( P_0 = 2 > C_0 = 1 \). Then, according to put/call parity, the put option is overvalued. Then you should follow the following arbitrage strategy:
Go long alternative 1: Write a put and buy a call

Cash flow now = $1
Cash flow at date 2 = $2 - $12.10

Go short alternative 2: Short the stock, collect $10 and invest the proceeds at \( r_2 = 10\% \) for two years.
Cash Flow now = 0
Cash flow at date 2 = 12.10 - $2

So total cash flow now = $1 and total cash flow at date 2 = 0, regardless of what $2 turns out to be. This is a riskless arbitrage. Everyone will try to short the stock (driving $0 down), write puts and buy calls until put/call parity holds.

Notice that put/call parity does not say that for the same \( X \) and \( T \), \( P_0 = C_0 \). However, in the special case where \( X = F_0 \), it must be the case that \( P_0 = C_0 \), as long as the cost of carry model holds. But this is nothing more than another way of saying that

You can replicate the long position in a forward contract by writing a put and buying a call with an exercise price equal to the forward price. Conversely, you can replicate the short position on a forward contract by writing a call and buying a put with an exercise price equal to the forward price.
Now that you are familiar with the basics of options, it is worthwhile to pause for a second and ask how investors can use options to achieve some desired objectives. The benefits of leverage when you use derivatives has already been discussed in the context of pure speculation, but there are some other interesting strategies that are worth a brief review.

Scenario: You think prices are going to be volatile but you are clueless which way they will move; up or down?
Strategy: Buy a "straddle". The easiest way to achieve a straddle position is to buy a put and a call option with the same exercise price and maturity date. In this case, if prices increase a lot, you can exercise the call, while if prices fall a lot you can exercise the put. If prices don't move too much, however, you are out both the price of the put and the price of the call.

Scenario: You already have a position in the underlying but want protect yourself against losses over some future period of time.

Strategy 1: "Protective Put". In this case you simply buy a put option with a maturity date equal to your investment horizon. If the price of the underlying increases, you are out the cost of the put but if it declines, your capital losses are offset by the gains from exercising the put.

Strategy 2: "Short Forward Position". In this case you agree to deliver however many units of the underlying that you own (ignoring dividends) at the forward price. This locks in your return, but that return must be the riskless spot rate if there is to be no arbitrage. This is an example of a general principle to be discussed later but what it says is essentially common sense.
If you fully hedge (insure) risk, the rate of return on your investment must equal the riskless rate.

Scenario: You have a position in some underlying but want to generate some current income by trading off future upside potential.

Strategy: "Covered Call Position". In this case you write a call option, which generates current income via the call price. If the value of the underlying increases in the future, the underlying gets "called away" from you and no gain on the underlying is realized, but if prices do not increase too much then you have your current income and the call remains unexercised.

General Properties of Option Prices:

The following are general properties of option prices that do not depend on specific models. Intuition is provided, although in some cases the answer is fairly obvious.

The price of a call, $C_0$, is
a. increasing in the current price of the underlying (increasing in $S_0$).
b. increasing in the interest rate (increasing in $rT$).
c. decreasing in the exercise price (decreasing in $X$).

Fact a. makes sense because the higher the current price, the greater the chance that the ultimate price will end up above the exercise price. Fact b. makes sense because an increase in the interest rate reduces the present value of the exercise price, which, according to fact c., makes the call option more
valuable. Finally, fact c. makes sense because a lower exercise price means that there is a greater chance that the call will end up in the money.

The price of a put, \( P_0 \), is
a. decreasing in the current price of the underlying (increasing in \( P_0 \)).
b. decreasing in the interest rate (decreasing in \( r_T \)).
c. increasing in the exercise price (increasing in \( X \)).

The intuition of the put relationships for these three variables is exactly the same as that for the call. The results are just the opposite. An increase in the underlying price or a decrease in the exercise price means there is less chance that the put will end up being profitable. Similarly, an increase in the interest rate causes a decrease in the present value of the exercise price, resulting in less chance that the put will end up making money for the holder.

The price of both puts and calls, \( P_0 \) and \( C_0 \), are
a. increasing in the time to maturity (increasing in \( T \)).
b. increasing in the volatility of the future stock price.

These two relationships make sense because a longer time to maturity gives the holder of either a put or a call "more chances" that his or her option will end up being exercised. Likewise, higher volatility means that there is a greater chance of both very high and very low prices. But the option holder is only going to exercise when it is in his or her interest, so the holder of a call likes the fact that very high prices are possible (and doesn't care about the low prices), while the holder of a put likes the fact that very low prices are possible (and doesn't care about the high prices).
Valuation of Options:

You already know, from put/call parity, that if you know the price of either the put or the call, you can determine the price of the other security, given the same exercise price and maturity on the two options. The lack of riskless arbitrage alone is generally not sufficient, however, to tell what you what the price of, say, the call option should be in the market. Another way to think of this is that some additional assumptions are needed to tell you exactly how the five factors discussed above interact to determine the price of the option given that there is to be no riskless arbitrage. For purposes of valuation, therefore, you need a specific model of how stock prices behave.

The two example models that are discussed in this section are:

a. the so-called "binomial" model, which assumes that the stock price can go either up or down by some percentage, not necessarily the same, over any one period of time and

b. the so-called Black/Scholes model, which assumes that the stock price changes continuously and, over very short intervals of time, the price changes have a normal distribution.

One way to think about this is to realize that if, for any fixed length of calendar time (think about from now until the option expires) the binomial has a lot of periods (make the periods short), then prices in case a. will be very close to those in case b., using the Black/Scholes model. While
Black/Scholes forms the basis of a lot of the "sophisticated" models used on Wall Street, the mathematics is very complex for a class like 4000. So the binomial model will be developed, in some detail, first. This will hopefully provide you with some sense of the important elements that go into more sophisticated formulas like Black/Scholes, which is covered after the binomial model. Finally, to keep things simple, the examples will generally cover cases of non-dividend paying securities.

**Binomial Option Pricing:**

The most efficient way to learn this approach to option pricing is to look at an example where the maturity of the option is two periods, which by convention in these notes is two years. Why two, you might ask. Well, once you understand how to solve the two period problem, you basically understand the "algorithm" that must be gone through to solve for option values in cases where the maturity is longer than two periods. On the flip side, one period problems are a breeze since, as you will see, you will have already solved two "one period" problems in order to find the value of the option with a maturity of two periods.

The basic approach to solving for the current arbitrage free value of an option involves starting at the end of the problem and working your way back to today; in this case starting at "date 1" and working your way back to "date 0" (today), all the time requiring that there be no riskless arbitrage over the next period, no matter how the stock price should move. Now comes some additional notation (@#!$%^&*(_+).
Given a time to maturity of $T (=2)$ and an interest rate $r_T (=r_2 = r_1) = r$, (assume the term structure is flat for now), assume that over any one period of time, one plus the rate of return on the stock (recall that this is the gross return) will be either "$u$" or "$d$", with $d < 1+r < u$, where $1+r$ is of course just the gross rate of return from holding the riskless security for one period. This means that, since the security pays no dividends, the price next period will be the price this period multiplied by either $u$ or $d$. This means that if the price at date 1 is $S_1$, then the price at date 2 will be either $S_2 = S_1u$ or $S_2 = S_1d$. But using the same reasoning, if $S_0$ is the current price of the security, then the price at date 1 must be either $S_1 = S_0u$ or $S_1 = S_0d$. So for the two period problem you now know what every possible security price and call price will be over the next two periods; Specifically, they will be

<table>
<thead>
<tr>
<th>Date</th>
<th>Price of Stock</th>
<th>Price of Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$S_0$</td>
<td>$C_0$</td>
</tr>
<tr>
<td>1</td>
<td>$S^+ = S_0u$ or $S^- = S_0d$</td>
<td>$C^+$ or $C^-$</td>
</tr>
<tr>
<td>2</td>
<td>$S^{++} = S^+ u = S_0u^2$ or $S^{-} = S_0du$</td>
<td>$C^{++}$ or $C^{+-}$</td>
</tr>
<tr>
<td></td>
<td>$or$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$S^{+-} = S_1d = S_0ud$ or $S^{--} = S_0d^2$</td>
<td>$C^{+-}$ or $C^{--}$</td>
</tr>
</tbody>
</table>

where, for example, $C^{+-}$ is the value of the call if the stock price first goes up at date 1 and then goes down at date 2. Notice also that $P_0du = P_0ud$, so there are really only three possible stock prices at date 2. Keeping in mind that the price of the option at maturity (date 2 in this case) is equal to it's
intrinsic value, the option price at date T will always be either 0 or ST - X, where X is the exercise price. Sometimes this will be written as CT = MAX(0, ST - X). For this problem T=2, so the value of the option at date 2 is C2 = MAX(0, S2 -X). Below are the general "steps" that you need to go through to solve for C0, the current price of the option.

One period example:
Before solving for the more complicated two period case, let's look at the problem when there is only one period to maturity (T=1). Here are the steps.

(A) One call option written.
(B) "H0" shares of the stock (H0 will turn out to be less than or equal to 1).
(C) Borrowing or lending at the risk free rate so that the net investment from (A)-(C) is zero. Next, choose H0 so that the cash flow from your position is the same whether the stock price goes up or down. In general, the cash flow will be

\[ -C_1 + H_0 S_1 + (C_0 - H_0 S_0)(1+r) \]

So if the stock price goes up, the cash flow will be

\[ -C^+ + H_0 S^+ + (C_0 - H_0 S_0)(1+r) \]

and if the stock price goes down the cash flow will be

\[ -C^- + H_0 S^- + (C_0 - H_0 S_0)(1+r) \]

Setting these two equations equal to one another and solving for H0 yields

\[ H_0 = (C^+ - C^-)/(S^+ - S^-) \]

By choosing the hedge ratio in this way the cash flow at date 1 is riskless. Moreover, the initial investment, as you will recall, is equal to zero. Therefore, the cash flow at date 1, regardless of whether the stock price goes up or down, must also be equal to zero. This is how we can solve for the option price at date 0. So the option price at date 0 is given by either

\[ C_0 = (C^+ - H_0 S^+)/ (1+r) + H_0 S_0 \]
Or

\[ C_0 = (C' - H_0 S') / (1+r) + H_0 S_0 \]

Either equation will work equally well and give the same answer.

Example:

Suppose that \( S_0 = $10, r = .05, X = $8, u = 1.5 \) and \( d = .5 \). Then \( S^+ = S_0(u) = 10(1.5) = 15 \) and \( S^- = S_0(d) = 10(.5) = 5 \). Moreover, \( C^+ = S^+ - X = 15 - 8 = 7 \) and \( C^- = 0 \), since the option is not exercised at a price of 5. Then the hedge ratio is given by \( H_0 = (C^+ - C^-) / (S^+ - S^-) = (7-0) / (15-5) = .7 \). Therefore the current price of the all option is given by \( C_0 = (C^+ - H_0 S^+)(1+r) + H_0 S_0 = (7-.7(15))(1.05) + .7(10) = $3.67. \)

Two period case:

It is possible, but a little more complicated, to extend this example to the two period case. Recall that at date 2, there can be one of three prices: \( S^{++}, S^{+-} = S^+ \) or \( S^- \), with corresponding option values of \( C^{++}, C^{+-} = C^+ \) or \( C^- \). So now the hedge ratio at date 1, \( H_1 \), can take on two values. The first is if the stock price goes up and is given by \( H_1^+ = (C^{++} - C^+) / (S^{++} - S^+) \) or \( H_1^- = (C^{+-} - C^-) / (S^{+-} - S^-) \). So using our earlier results, we can find the values of \( C^+ \) and \( C^- \) and then work our way backwards to find \( C_0 \). In particular, we have that \( C^+ = (C^{++} - H_1^+ S^{++}) / (1+r) + H_1^+ S^+ = (C^+ - H_1^+ S^+) / (1+r) + H_1^+ S^+ \) and \( C^- = (C^{+-} - H_1^- S^{+-}) / (1+r) + H_1^- S^- = (C^- - H_1^- S^-) / (1+r) + H_1^- S^- \). Finally, we can use the earlier formulas to calculate \( C_0 \).

Example:

Let’s continue with the example we used earlier. In this case we have that \( S^{++} = S^+ u = 15(1.5) = 22.5, S^{+-} = S^- u = 5(1.5) = 7.5 = S^+ = S^+ d= 15(.5) = 7.5 \) and \( S^- = S^- d = 5(.5) = 2.5 \). Similarly, we have that \( C^{++} = 22.5 - 8 = 14.5, \) and \( C^{+-} = C^+ = C^- = 0 \), since the option will not be exercised in any of these
cases. So $H_1^+ = (C^{++} - C^+)/ (S^{++} - S^+) = (14.5 - 0)/(22.5 - 7.5) = .9667$, while
$H_1^- = (C^+ - C^-)/(S^+ - S^-) = (0-0)/(7.5-2.5) = 0$. So,
$C^+ = (C^{++} - H_1^+ S^{++})/(1+r) + H_1^+ S^+ = (14.5 - .9667(22.5)) / (1.05) + .9667(15) = (C^+ - H_1^+ S^+)/ (1+r) + H_1^+ S^+ = (0 - .9667(7.5)) / (1.05) + .9667(15) = 7.595.$
$C^- = (C^{++} - H_1^- S^{++})/(1+r) + H_1^- S^- = (0-0)/(1.05) + (0)(5) = 0$. Finally, we have
that $H_0 = (7.595 - 0)/(15-5) = .7595$
$C_0 = (C^+ - H_0 S^+)/ (1+r) + H_0 S_0 = (7.595-.7595(15)) / (1.05) + .7595(10) = (C^- - H_0 S^-)/(1+r) + H_0 S_0 = (0 - .7595(5)) / (1.05) + .7595(10) = 3.9783.$