Management Compensation and Market Timing under Portfolio Constraints*

Vikas Agarwal†, Juan-Pedro Gómez‡ and Richard Priestley§

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Abstract

We analyze the implications of short-selling and margin purchase constraints for portfolio management compensation and market timing ability under moral hazard. We solve analytically for the benchmark composition that maximizes effort expenditure under portfolio constraints. Studying the principal’s optimal contract, we show that, absent portfolio constraints, relative performance evaluation is suboptimal and the incentives for effort expenditure (hence, market timing ability) are maximized. Under portfolio constraints, effort incentives decrease and relative performance evaluation may be optimal. Numerically, we solve jointly for the manager’s incentive fee and the optimal benchmark. We show that choosing the benchmark that maximizes the fund’s Information Ratio may be suboptimal for the fund investors.

Keywords: Market Timing, Incentive Fee, Benchmarking, Portfolio Constraints

JEL Classification Numbers: D81, D82, J33.

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†J. Mack Robinson College of Business, Georgia State University. E-mail: vagarwal@gsu.edu

‡Corresponding author, Instituto de Empresa, Castellón de la Plana 8, 28006 Madrid, Spain. Phone: +34 91 782 1326. Fax: +34 91 745 4762. E-mail: juanp.gomez@ie.edu

§Department of Financial Economics, Norwegian School of Management, Norway. E-mail: richard.priestley@bi.no
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1 Introduction

In this paper, we study the effect of relative (to a benchmark) performance evaluation on the provision of incentives for the search of private information under moral hazard when managers face exogenous portfolio constraints that limit their ability to sell short and purchase on margin.

Portfolio constraints have not been examined in great details in the extant literature on portfolio delegation.\footnote{Almazan, Brown, Carlson and Chapman (2004) document that approximately 70% of mutual funds explicitly state (in Form N-SAR submitted to the SEC) that short-selling is not permitted. This figure rises to above 90% when the restriction is on margin purchases. They present evidence that portfolio constraints are devices to monitor the manager’s effort. Grinblatt and Titman (1989) and Brown, Harlow, and Starks (1996) argue that cross-sectional differences in constraint adoption might be related to characteristics that proxy for managerial risk aversion.} This paper shows that portfolio constraints have important implications for management compensation and performance evaluation. Specifically, our paper makes three contributions. First, taking portfolio constraints as given, we solve analytically for the benchmark composition that maximizes the manager’s effort expenditure. Second, analyzing the principal’s optimal contract, we show that, under portfolio constraints, relative performance evaluation may be optimal. Numerically, we solve jointly for the manager’s incentive fee and the optimal benchmark. Third, under portfolio constraints, when the benchmark composition is endogenously determined, the principal’s optimal benchmark choice will not necessarily coincide with the benchmark that maximizes the fund’s Information Ratio (excess return per unit of tracking error volatility).

We propose a two-period, two-asset (the market and a risk-less bond) model. Principal in our model is the investor who plays the role of a fund’s management company. The compensation contract is signed between the fund’s management company and the portfolio manager. This contract is usually unobservable.\footnote{Typically, the fund’s board will inform investors that managers (who are involved in investment research) are responsible for choosing the fund’s investments. Investors are also informed about how the advisory management company (responsible for choosing and monitoring the managers) is compensated. This is known as the advisory fee. Given this information, the investors decide how much to invest in the fund. In this paper, we abstract from the decision problem of the investor and the relationship between the fund’s board and the management company. For recent empirical studies of fund advisory fees see, for instance, Deli (2002), Warner and Wu (2005), and Massa and Patgiri (2008).} The compensation package includes a flat fee and a performance based incentive fee, possibly benchmarked to a given portfolio return. Both the incentive fee and the benchmark composition are determined endogenously. In our model, the manager’s incentives are explicit: they arise from the design of the optimal compensation contract.\footnote{In the model, the fund’s net asset value is given. Both the flat fee and the incentive fee are proportional to the fund’s net asset value. We abstract from the \textit{implicit incentives} arising from the convex flow-performance relation documented in the literature (see, for instance, Gruber (1996), Sirri and Tufano (1998), Chevalier and Ellison (1997), Del Guercio and Tkac (2000), and Basak, Pavlova, and Shapiro (2007)).}

A number of new insights arise after introducing portfolio constraints. First, looking at the manager’s effort and portfolio choice problem, we show that the active portfolio and effort decisions (hence, performance) depend on both the incentive fee and the benchmark composition. The relationship between the manager’s effort and the incentive fee has been studied by Gómez and Sharma (2006). The relationship between the effort decision and the bench-
mark composition, however, contrasts with the well-known “irrelevance result” in Admati and Pfeiderer (1997): the manager’s effort is independent of the benchmark composition; it only depends on the manager’s effort disutility. We derive explicitly the effort maximizing benchmark’s composition as a function of the market moments, the portfolio constraints, and the manager’s risk-aversion coefficient. We show that the irrelevance result in Admati and Pfeiderer (1997) arises only in the limit, when there are no portfolio constraints.

To understand the model’s intuition, consider a manager who is totally constrained in her ability to sell short and purchase at margin. Under moral hazard, the manager’s optimal portfolio can be decomposed in two components: her unconditional risk-diversification portfolio plus her active or “timing” portfolio. The timing portfolio depends on the manager’s costly effort to improve her timing ability through superior information. For an uninformed manager, this portfolio would be zero. For a hypothetical perfectly informed manager, it would push the optimal total portfolio to either boundary: 100% in the risky asset if the market risk premium is forecasted to be positive; 100% in the bond otherwise. Now, assume that the unconditional risk-diversification portfolio consists of 30% invested in the risky market portfolio. For this perfectly informed manager, any timing portfolio that involves shorting the market by more than 30% or investing more than 70% in the market will hit the portfolio boundaries. Anticipating this and taking into account effort disutility, the manager will decide the optimal effort expenditure.

Imagine now that the same manager is given a benchmarked contract. The benchmark consists of 20% in the market portfolio and 80% in the bond. The manager adjusts her optimal portfolio. Relative to the benchmark, the unconditional optimal risk-diversification portfolio is still 30% long in the market. The manager has to beat the benchmark for the incentive fee to kick in. Therefore, her total market investment will be now 50% of her portfolio: 20% to replicate the benchmark plus 30% for the optimal risk-diversification. Holding the portfolio constraints constant, this implies that if the market premium is predicted to be negative, the manager’s timing portfolio can now go short up to 50% in the market, 20% more than in the absence of the benchmark. This will increase the manager’s utility from effort, thereby improving the incentives for sharpening her timing ability. At the same time, if the market premium is predicted to be positive, the manager’s timing portfolio can go long in the market only 50%, 20% less than before the benchmark was introduced. This has the opposite effect on the effort inducement: the manager will have less incentive to exert costly effort. Taking into account this trade-off, the benchmark is chosen such that the manager’s unconditional portfolio (benchmark replication plus optimal risk-return trade-off) is equidistant from both portfolio boundaries. Such a benchmark would provide the manager with the highest incentives for effort exertion.

The intuition is simple: such a benchmark leaves the manager marginally indifferent between hitting the short-selling constraint or the margin purchase constraint. When the portfolio space is unconstrained, so is the timing portfolio. Benchmarking the manager’s incentive fee fails to provide any incentive for effort expenditure.

Turning to the investor’s problem, he has to decide the benchmark composition and the fee structure. We obtain two conclusions. First, we show that in the absence of moral hazard

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4Since the manager needs to optimally choose between a well-diversified market portfolio and the risk free asset, active management in our model is analogous to timing ability. Hence, we use the two terms interchangeably.
between the investor and the fund manager, the optimal incentive fee coincides with the Pareto-efficient risk allocation fee. In addition, we show that the optimal benchmark is the risk-free asset. Since the return on the risk-free asset is known with certainty, it is equivalent to absolute performance evaluation. This is not totally surprising: in the absence of moral hazard, the manager’s effort is independent of the incentive fee and the benchmark composition. The only role for the incentive fee is to split the risk between the investor (the principal) and the manager (the agent). Hence, the first best split remains optimal. As for the benchmark, any deviation from the risk-free asset (uncorrelated with the market portfolio) will distort the principal’s optimal portfolio and induce the manager to take excessive risk above the investor’s risk preference (Roll (1992)). This result extends the unconstrained contract of Adamati and Pfeiderer (1997) into the constrained scenario: with or without constraints, the investor’s optimal benchmark when effort is publicly observable is the minimum-variance portfolio (in this case, the risk free asset).

This does not necessarily hold in the presence of moral hazard between the investor and the manager. Under portfolio constraints and moral hazard, the manager’s effort depends on the incentive fee and the benchmark composition. On the one hand, increasing the incentive fee gives the manager more incentives to improve her timing ability (by putting more effort); the downside is that the compensation becomes more onerous for the investor. With respect to the benchmark, the risk free asset may not be optimal anymore: making the benchmark more risky may induce higher greater effort on the manager. On the other hand, any benchmark other than the risk-free asset will affect the investor’s optimal risk-return trade-off. Moreover, these double trade-off considerations (for the incentive fee as well as for the benchmark composition), are interrelated.

We show analytically that, in the presence of moral hazard and portfolio constraints, the incentive fee contract under no moral hazard is suboptimal. Numerical results show that the optimal incentive fee is higher than in the case of no moral hazard. Moreover, contrary to the unconstrained case in Adamati and Pfeiderer (1997), the optimal benchmark is different from the risk-free asset and varies with the portfolio constraints. More concretely, the optimal benchmark proportion invested in the market increases with the manager’s risk aversion and decreases with the investor’s risk aversion. From the investor’s point of view, providing effort incentives is not the only concern for the investor. Given the benchmark portfolio, the manager will maximize the excess relative return while reducing the tracking error volatility. In other words, the manager will try to maximize the fund’s Information Ratio (relative excess return per unit of tracking error). This comes, however, at the expense of exposing the investor to higher total risk. We show that when the benchmark composition is endogenously determined, maximizing the fund’s Information Ratio is generally suboptimal for the principal. In other words, the principal’s optimal benchmark choice will not necessarily coincide with the benchmark that maximizes the fund’s Information Ratio. Only when the manager is, relative to the principal, sufficiently risk-averse, maximizing the Information Ratio becomes optimal for the investor.

The rest of the paper is organized as follows. Next we review the related literature. Section 3 introduces the model. The standard unconstrained results are reviewed in Section 3.1 while the effect of portfolio constraints are analyzed in Section 3.2. In Section 4, we derive the composition of the effort-maximizing benchmark portfolio. Section 5 studies the principal’s
problem. A numerical solution to the constrained manager’s effort is presented in Section 6. Section 7 introduces stock picking ability in the model and Section 8 concludes the paper. All proofs are presented in the Appendix.

2 Related Literature

The design of fund management compensation schemes has elicited interest amongst both practitioners and researchers. The focus of the academic literature has been on how incentives affect performance and risk-taking behavior of managers. A number of theoretical papers have studied the effect of a performance-related incentive fee on managers’ incentive to search for private information (see, for example, Bhattacharya and Pfleiderer (1985), Stoughton (1993), Heinkel and Stoughton (1994) and Gómez and Sharma (2006)). Another strand of literature addresses issues related to the design of incentive fee. Adamati and Pfleiderer (1997) and Dybvig, Farnsworth and Carpenter (2009), among others, have discussed the convenience of absolute versus relative (benchmarked to a given portfolio) incentive fees.\footnote{A further line of discussion concerns whether, if benchmarked, the incentive fee should be “convex” (i.e. asymmetric), implying that the manager only participates in the upside and suffers no penalty for underperforming the benchmark, or, as prescribed by the Securities and Exchange Commission (SEC) for mutual funds, a “fulcrum” (symmetric) type of fee. See, for example, Starks (1987), Das and Sundaram (2002) and Ou-Yang (2003).}

With respect to risk, Roll (1992) was the first to illustrate the undesirable effect of relative (i.e., benchmarked) portfolio optimization in a partial equilibrium, single-period model. In particular, he shows that the manager’s active portfolio is independent of the benchmark composition and that this leads the manager to take systematically more risk than the benchmark. Despite the sub-optimal risk allocation, the portfolio optimization literature takes as given that the manager minimizes tracking error volatility subject to an excess return and studies how different constraints on the portfolio’s total risk (Roll (1992)), tracking error (Jorion (2003)), and Value-at-Risk (VaR) (Alexander and Baptista (2008)), may help to reduce excessive risk taking. Bajeux et al (2007) study the interaction between tracking error and portfolio weight constraints. Interestingly, Jorion (2003) writes (footnote 7, page 82): “in practice, the active positions will depend on the benchmark if the mandate has short-selling restrictions on total weights.” Our model formalizes this intuition and shows that, in the presence of portfolio constraints, the manager’s active portfolio depends on the benchmark composition. More importantly, for the constrained manager, the tracking-error minimization mandate arises endogenously through the manager’s relative incentive fee.

In a dynamic setting, Basak, Shapiro, and Tepla (2006) study the optimal policies of an agent subject to a benchmarking restriction. Basak, Pavlova, and Shapiro (2008) analyze the effect of an exogenous benchmark restriction on the manager’s risk-taking behavior. Their model shows that an exogenous benchmark restriction may ameliorate the adverse risk incentives induced by the manager’s compensation as a proportion of the fund’s net asset value (implicitly linked to the fund’s performance). In a multi-manager portfolio delegation problem, Binsbergen, Brandt, and Koijen (2007) show that benchmarking the managers payoffs may help to align the interest of the investor and the managers for risk diversification, risk sharing and investment horizons.
However, they do not study the optimal benchmark composition for effort inducement: in their model, the manager’s timing ability is exogenously given. Brennan (1993), Gómez and Zapatero (2003) and Cuoco and Kaniel (2006) study the asset pricing implications of relative incentive fees.

Finally, our paper is also related to the literature on mutual fund performance evaluation and, more specifically, the evidence on the market timing ability across of mutual funds: Treynor and Mazuy (1966), Henriksson and Merton (1981), Jagannathan and Korajczyk (1986), Becker, Ferson, and Schill (1999), Goetzmann, Ingersoll, and Ivkovich (2000), Bollen and Busse (2001), Jiang (2003), Jiang, Yao, and Yu (2007), among others. Our model shows how to extend the tests in Becker et al. (1999) into a framework that accounts explicitly for the presence of short selling and margin purchase constraints, prevalent across the mutual fund industry.

3 The model

The manager and the investor have preferences represented by exponential utility functions: $U_a(W) = -\exp(-aW)$ and $U_b(W) = -\exp(-bW)$, respectively. Throughout the paper, we will use $a > 0$ ($b > 0$) to denote the manager (investor) as well as her (his) absolute risk aversion coefficient. The investment opportunity set consists of two assets: a risk-free asset with gross return $R$ and a stock with stochastic excess return $x$ normally distributed with mean excess return $\mu > 0$ and volatility $\sigma$. These two assets can be interpreted as the usual “timing portfolios” for the active manager: the bond and the market portfolio (or any other stochastic timing portfolio).

The investment horizon is one period. Payoffs are expressed in units of the economy’s only consumption good. All consumption takes place at the end of the period. The manager’s compensation is set as a percentage of the fund’s average net asset value over the period, $W$. The percentage has two components: a fixed basic fee $F$ and an incentive (performance-based) fee. The incentive fee is calculated as a percentage $\alpha \in (0, 1]$ of the fund’s end-of-the-period return, possibly net of a benchmark return.

After learning the contract, the manager decides whether to accept it or not. If rejected, the manager gets her reservation value. If she accepts the contract, then she puts some (unobservable) effort $e > 0$ in acquiring private information (not observed by the fund’s investor) that comes in the form of a signal

$$y = x + \frac{\sigma}{\sqrt{\epsilon}} e,$$

partially correlated with the stock’s excess return. The noise term has a standard normal distribution $\epsilon \sim N(0, 1)$. For simplicity, we assume

**Assumption (S1)** $E(x\epsilon) = 0$.

The greater the effort the more precise the manager’s timing information. Conditional on the manager’s effort, the stock’s excess return is normally distributed with conditional mean return $E(x|y) = \frac{\mu + ey}{1+e}$ and conditional precision $\text{Var}^{-1}(x|y) = \frac{1}{\sigma^2}(1 + e)$. Hence, $e$ can also be interpreted as the percentage (net) increase in precision induced by the manager’s private
information. Notice that, in case \( e = 0 \), the conditional and unconditional distributions coincide: there is no relevant private information.

Effort is costly. The monetary cost of effort disutility is a percentage \( V(D, e) \) of the fund’s net asset value \( W \). \( D > 0 \) represents a disutility parameter. The function \( V \) is increasing in \( D \) and homogenous of degree one with respect to \( D \). Moreover, for all \( e > 0 \), \( V \) satisfies:

\[ V(D,0) = V_e(D,0) = V(0,e) = 0, \]

**Assumption (S2)** \( V(D,0) = V_e(D,0) = V(0,e) = 0, \)

**Assumption (S3)** \( V_e(D,e) > 0, \)

**Assumption (S4)** \( \frac{V_{ee}(D,e)}{V_e(D,e)} > \frac{1}{1+e}. \)

### 3.1 Unconstrained Portfolio Choice

Based on the conditional moments, the manager makes her optimal portfolio decision: she will invest a percentage \( \theta(y) \) in the stock and the remaining \( 1 - \theta(y) \) in the risk-free bond. Therefore, the portfolio’s return will be \( R_p = R + \theta x \). Define the benchmark’s return as \( R_h = R + hx \) with \( h \) as the *benchmark’s policy weight*: the proportion in the benchmark portfolio invested in the risky stock. The portfolio’s *net return* is given by \( R_p - R_h = \bar{\theta}x \) with \( \bar{\theta} = \theta - h \), the net (over the benchmark) investment in the risky stock. If \( h = 0 \), the benchmarked return is \( R_p - R_h = \theta x \), the excess return. Since the risk-free return is a constant, from the point of view of the manager, this case is equivalent to no benchmarking.

Given a contract \((F, \alpha, h)\), the conditional end-of-the-period wealth is given as a percentage \( \varphi_a \), for the manager, and \( \varphi_b \), for the investor, of the fund’s net asset value, \( W \):

\[
\begin{align*}
\varphi_a(\bar{\theta}) &= F + \alpha \bar{\theta} x, \\
\varphi_b(\bar{\theta}) &= R_h + (1 - \alpha) \bar{\theta} x - F,
\end{align*}
\]

with \( \bar{\theta} = \bar{\theta}(y) \) and \( x = x(y) \), functions of the signal realization \( y \). If the manager chooses the benchmark portfolio as her optimal portfolio then \( \bar{\theta} = 0 \); the manager receives no incentive fee (only the fixed fee \( F \)) and the investor’s payoff is the benchmark’s return net of the fixed fee.\(^7\)

After these definitions, the conditional utility function for the manager and the investor can be expressed, respectively, as

\[
\begin{align*}
U_a (\varphi_a(\bar{\theta})) &= -\exp (-a \varphi_a(\bar{\theta}) W + V(D, e) W), \\
U_b (\varphi_b(\bar{\theta})) &= -\exp (-b \varphi_b(\bar{\theta}) W).
\end{align*}
\]

In this setting, the Arrow-Pratt risk premium for the manager will be, \( \alpha W \frac{a W}{2} \bar{\theta}^2 \sigma^2 \). Thus, \( aa W \) represents the manager’s *relative* risk aversion coefficient. For simplicity, and without loss \(^6\) the subscripts \( e \) and \( ee \) denote, respectively, first and second derivative with respect to effort.

\(^7\) Sometimes the benchmark may include a minimum excess return \( \tau > 0 \) such that \( R_h = R + \tau + hx \). Notice that this is equivalent to defining \( F = F' - \alpha \tau \) in equations (1) and (2). Solving for \( F \) and \( \alpha \), \( F' \) is obtained as a function of \( \tau \).
of generality, we normalize $W = 1$.

We shall proceed backwards. First, we will obtain the optimal portfolio choice $\theta$. Then, after recovering the manager’s indirect utility function, we will tackle the manager’s effort decision. The unconstrained manager’s optimal net portfolio solves

$$\tilde{\theta}(y) = \arg \max_\theta \{ E(\varphi_\theta(\tilde{\theta})) - (a/2)\Var(\varphi_\theta(\tilde{\theta})) \},$$

which yields the optimal portfolio

$$\theta(y) = h + \frac{\mu}{a\alpha \sigma^2} + \frac{ey}{a\alpha \sigma^2}.$$

(3)

The manager’s optimal portfolio has three components: the benchmark’s investment in the risky stock, $h$; the unconditional optimal risk-return trade-off, $\frac{\mu}{a\alpha \sigma^2}$ and, depending on the manager’s signal $y$ and her effort expenditure, $e$, the timing portfolio, $\frac{ey}{a\alpha \sigma^2}$. Replacing $\theta(y)$ in the manager’s expected utility function and integrating over the signal $y$ we obtain the manager’s (unconditional) expected utility:

$$EU(\varphi_\theta(e)) = -\exp \left( -(1/2)(\mu^2/\sigma^2) - aF + V(D, e) \right) g(e),$$

with $g(e) = \left( 1 + e \right)^{1/2}$. At the optimum, the marginal utility of effort must be equal (first-order condition) to its marginal disutility:

$$V_e(D, e_{SB}) = \frac{1}{2(1 + e_{SB})}.$$  

(5)

We call this solution the second best effort. Assumptions (S2) and (S3) guarantee that the necessary condition (5) is also sufficient for optimality. Clearly, the manager’s second best effort choice (hence the quality of her private information) is independent of the benchmark’s composition, $h$. This is the same result as in Admati and Pfleiderer (1997). Effort only depends on the manager’s disutility coefficient, $D$.

### 3.2 Constrained Portfolio Choice

We now introduce one of the main contributions in the paper. Assume that the manager is constrained in her portfolio choice in that she cannot short-sell or purchase on margin. Let $m \geq 1$ denote the maximum trade on margin the manager is allowed: $m = 1$ means that the manager is not allowed to purchase the risky stock on margin; for any $m > 1$ the manager can borrow and invest in the risky stock up to $m - 1$ dollars per dollar of the fund’s current net asset value. Let $s \geq 0$ denote the short-selling limit: $s = 0$ means that the manager cannot sell short the risky stock; for any $s > 0$ the manager can short up to $s$ dollars per dollar of the fund’s current net asset value. According to the SEC regulation, the maximum initial margin for leveraged positions is 50%, which implies that $m \leq 2$ and $s \leq 1$. Of course, investors can effectively leverage their portfolios above those limits by investing in derivatives.
portfolio choice problem, this implies \( m \geq \theta \geq -s \) or, equivalently, \( m - h \geq \tilde{\theta} \geq -(h + s) \).

The manager then solves the following constrained problem

\[
\hat{\theta}(y) = \arg \max_{m-h \geq \theta \geq -(h+s)} \left\{ E(\varphi_a(\theta)) - (a/2)\text{Var}(\varphi_a(\theta)) \right\}.
\]

Call \( \lambda_m \leq 0 \) and \( \lambda_s \leq 0 \) the corresponding Lagrange multipliers, such that \( \lambda_m(m - h - \tilde{\theta}) = \lambda_s(\tilde{\theta} + h + s) = 0 \). There are three solutions. If neither constraint is binding, \( \lambda_m = \lambda_s = 0 \), then the interior solution follows: \( \hat{\theta}(y) = \frac{\mu + ey}{a \alpha \sigma^2} \). Alternatively, there are two possible corner solutions: first, if the short-selling limit is binding, \( \lambda_m = 0 \) and \( \lambda_s = E(x|y) + a\alpha(h + s)\text{Var}(x|y) < 0 \). In such a case, \( \hat{\theta} = -(h + s) \). In the second corner solution, the margin purchase bound is hit: \( \lambda_s = 0 \) and \( \lambda_m = -E(x|y) + a\alpha(m - h)\text{Var}(x|y) < 0 \). In such a case, \( \hat{\theta} = m - h \).

Solving for the optimal portfolio \( \theta(y) \) as a function of the signal realization we obtain that, in the case of no timing ability \( (e = 0) \), \( \theta = h + \frac{\mu}{a \alpha \sigma^2} \) provided \( - (s + \frac{\mu}{a \alpha \sigma^2}) \leq h \leq m - \frac{\mu}{a \alpha \sigma^2} \).

For the case when \( e > 0 \) we obtain:

\[
\theta(y) = \begin{cases} 
-s & \text{if } y < -\frac{\mu}{e}L_s \\
\frac{\mu}{a \alpha \sigma^2} + \frac{ey}{a \alpha \sigma^2} & \text{otherwise} \\
m & \text{if } y > \frac{\mu}{e}L_m 
\end{cases}
\]

We call

\[
L_s(h) = 1 + (h + s) \left( \frac{\mu}{a \alpha \sigma^2} \right)^{-1}
\]

\[
L_m(h) = (m - h) \left( \frac{\mu}{a \alpha \sigma^2} \right)^{-1} - 1
\]

the leverage ratios. These ratios represent the net (relative to the benchmark) maximum leverage from selling short \((h + s)\) or trading at margin \((m - h)\) as a proportion of the manager’s optimal unconstrained portfolio when \( e = 0 \) and \( h = 0 \).

Looking at the way the leverage ratios change with benchmarking, we observe that \( \frac{\partial}{\partial h} L_s = \left( \frac{\mu}{a \alpha \sigma^2} \right)^{-1} > 0 \) and \( \frac{\partial}{\partial m} L_m = - \left( \frac{\mu}{a \alpha \sigma^2} \right)^{-1} < 0 \). That is, \( L_s \) \((L_m)\) increases (decreases) with \( h \). Moreover, given the (risk-adjusted) market premium \( \mu/\sigma^2 \), the marginal change in \( L_s \) \((L_m)\) increases (decreases) with the manager’s relative risk aversion \( a\alpha \).

Equation (6) shows how the constraints and benchmarking interact to provide incentives for effort expenditure. To see the intuition, let us focus first on the short-selling constraint. Let us assume for the moment that there exists no limit to margin purchases \((m \to \infty)\) and that no short position can be taken \((s = 0)\). Under these assumptions, and after exerting effort \( e \), the manager receives a signal \( y \) and makes her optimal portfolio choice:

\[
\theta(y) = \begin{cases} 
0 & \text{if } y < -\frac{\mu}{e}L_s \\
h + \frac{\mu + ey}{a \alpha \sigma^2} & \text{otherwise,}
\end{cases}
\]
with $L_s = 1 + h \left( \frac{\mu}{\alpha \sigma^2} \right)^{-1}$. When $h = 0$, all signals $y < -\frac{\mu}{\sigma}$ lead to short-selling. Imagine now that the manager is offered a benchmarked contract, with $h > 0$ the benchmark’s proportion invested in the risky stock. In this case, the short-selling bound is only hit for smaller signals $y < -\frac{\mu}{\sigma} L_s$. In general, increasing $h$ leads to a “wider range” of implementable signals relative to the case of no benchmarking ($h = 0$). Since the effort decision is taken prior to the signal realization, the fact that more signals are implementable under benchmarking ($h > 0$) increases the marginal expected utility of effort. The size of this incremental area grows with $\alpha \sigma$. Hence, we expect the impact of benchmarking to be relatively higher for more risk averse investors.

Alternatively, assume there is no benchmarking ($h = 0$) but the short-selling limit is expanded from $s = 0$ to $s = h$. Figure 1 shows that, ceteris paribus, the effort choice of the manager will coincide with the effort put under benchmarking: given that $s = 0$, benchmarking the manager’s portfolio return ($h > 0$) is, in terms of effort inducement, equivalent to relaxing the short-selling bound from 0 to $h$. In other words, in the absence of margin purchase constraints, the manager’s effort depends on $s + h$; benchmarking the manager’s performance and relaxing her short-selling constraints are perfect substitutes for effort inducement. The higher is $s$, the lower the marginal expected utility of effort induced by benchmarking. In the limit, when the short-selling bounds vanish ($s \to \infty$), we converge to the unconstrained scenario in Section 3.1 where benchmarking was shown to be irrelevant for the manager’s effort decision.

Let us now focus on the margin purchase constraint. Assume $s \to \infty$ and $m = 1$. This implies that the manager can short any amount but cannot trade on margin: for “very good” signals the manager can only invest up to 100% of the fund’s net asset value in the risky stock. Her optimal portfolio (as a function of the signal) will be:

$$\theta(y) = \begin{cases} 
1 & \text{if } y > \frac{\mu}{\sigma} L_m, \\
\frac{1}{h + \frac{\mu + \epsilon y}{\alpha \sigma^2}} & \text{otherwise},
\end{cases}$$

with $L_m = (1 - h) \left( \frac{\mu}{\alpha \sigma^2} \right)^{-1} - 1$. $L_m$ is decreasing in $h$. Decreasing $h$ in the manager’s compensation just makes the portfolio constraint “less binding,” i.e., binding only for bigger signals. For instance, moving from a benchmarked contract ($h > 0$) to a non-benchmarked contract ($h = 0$) would increase the manager’s effort: signals that were not implementable under benchmarking become now feasible. Symmetric to the short-selling constraint, the expected impact on effort expenditure would be analogous if benchmarking were not removed ($h > 0$) and the constraint on margin purchases made looser: from $m = 1$ to $m = 1 + h$. Therefore, in the absence of short selling constraints, the manager’s effort depends on $m - h$: benchmarking the manager and tightening the margin purchase constraint are perfect substitutes for the manager’s effort (dis)incentive. Again, the impact of benchmarking increases, in absolute terms, with the manager’s relative risk aversion, $\alpha \sigma$. In the limit, when the manager faces no margin purchase constraint ($m \to \infty$), the benchmark composition is irrelevant for the manager’s effort decision.

In summary, by modifying the benchmark portfolio composition we observe two opposing effects: for the short selling constrained manager, increasing the benchmark’s percentage invested in the risky stock ($h$) induces the manager to put more effort. In contrast, for the manager
constrained in her ability to purchases at margin, increasing that percentage lowers the effort incentives. Thus, when (as for most mutual fund managers) both short selling and margin purchase are constrained, the trade-off between these two effects yields the effort-maximizing benchmark. This is the question we investigate in the next section.

4 The effort-maximizing benchmark

To address the question of what is the composition of the effort-maximizing benchmark, we proceed as follows. Proposition 1 introduces the manager’s unconditional expected utility under short selling \((0 \leq s < \infty)\) and margin purchase \((1 \leq m < \infty)\) constraints for all possible values of \(h\) in the real line. In Proposition 2, we show that Assumptions (S2)-(S4) are sufficient for the existence of a continuous and differentiable effort function, \(e(h)\), that yields a unique effort choice for each value of \(h\). The function attains a global maximum at \(h^* = \frac{m-s}{2} - \frac{\mu}{\omega\sigma^2}\).

Before introducing the constrained manager’s unconditional expected utility we need some notation. Let \(\Phi(\cdot)\) denote the cumulative probability function of a Chi-square variable with one degree of freedom: \(\Phi(x) = \int_0^x \phi(z) \, dz\), with

\[
\phi(z) = \begin{cases} \frac{1}{\sqrt{2\pi}} z^{-1/2} \exp(-z/2) & \text{when } z > 0; \\ 0 & \text{otherwise.} \end{cases}
\]

Proposition 1 Given the finite portfolio constraints \(s \geq 0\) and \(m \geq 1\), the risk-averse manager’s expected utility is

\[
EU_a(\varphi_a(e)) = -\exp(-(1/2)\mu^2/\sigma^2 - \alpha F + V(D, e)) \times g(e, L_s, L_m) \quad \text{with} \quad g(e, L_s, L_m) = (1/2) \times
\]

\[
\exp\left(\frac{\left(\frac{\mu}{\sigma} L_s\right)^2}{2}\right) \left[1 + \Phi\left(\frac{1+e}{e} \left(\frac{\mu}{\sigma} L_s\right)^2\right)\right] + \\
\left(\frac{1}{1+e}\right)^{1/2} \left[\Phi\left(\frac{\left(\frac{\mu}{\sigma} L_m\right)^2}{e}\right) - \Phi\left(\frac{\left(\frac{\mu}{\sigma} L_s\right)^2}{e}\right)\right] + (7)
\]

\[
\exp\left(\frac{\left(\frac{\mu}{\sigma} L_m\right)^2}{2}\right) \left[1 - \Phi\left(\frac{1+e}{e} \left(\frac{\mu}{\sigma} L_m\right)^2\right)\right]
\]

if \(h < -(s + \frac{\mu}{\omega\sigma^2})\); we have

\[
\exp\left(\frac{\left(\frac{\mu}{\sigma} L_s\right)^2}{2}\right) \left[1 - \Phi\left(\frac{1+e}{e} \left(\frac{\mu}{\sigma} L_s\right)^2\right)\right] + \\
\left(\frac{1}{1+e}\right)^{1/2} \left[\Phi\left(\frac{\left(\frac{\mu}{\sigma} L_s\right)^2}{e}\right) + \Phi\left(\frac{\left(\frac{\mu}{\sigma} L_m\right)^2}{e}\right)\right] + (8)
\]

\[
\exp\left(\frac{\left(\frac{\mu}{\sigma} L_m\right)^2}{2}\right) \left[1 - \Phi\left(\frac{1+e}{e} \left(\frac{\mu}{\sigma} L_m\right)^2\right)\right]
\]
\[ \text{if } (s + \frac{\mu}{\alpha \sigma^2}) \leq h \leq m - \frac{\mu}{\alpha \sigma^2}; \]

\[ \exp \left( \frac{(\frac{\mu}{\alpha} L_s)^2}{2} \right) \left[ 1 - \Phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\alpha} L_s \right)^2 \right) \right] + \]

\[ \left( \frac{1}{1 + e} \right)^{1/2} \left[ \Phi \left( \frac{(\frac{\mu}{\alpha} L_s)^2}{e} \right) - \Phi \left( \frac{(\frac{\mu}{\alpha} L_m)^2}{e} \right) \right] + \]

\[ \exp \left( \frac{(\frac{\mu}{\alpha} L_m)^2}{2} \right) \left[ 1 + \Phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\alpha} L_m \right)^2 \right) \right] \]

\[ \text{if } h > m - \frac{\mu}{\alpha \sigma^2}. \]

Equations (7), (8) and (9) are weighted sums of the manager’s unconstrained expected utility (4), independent of \( h \), and her expected utility function when the portfolio hits either the short-selling constraint bound, \( \exp \left( \frac{(\frac{\mu}{\alpha} L_s)^2}{2} \right) \), or the margin purchase bound, \( \exp \left( \frac{(\frac{\mu}{\alpha} L_m)^2}{2} \right) \). When the manager is constrained, the benchmark’s composition (i.e., the value of the parameter \( h \)) affects the quality of the timing signal through the effort choice.

**Corollary 1** The first derivative \( g_e(e, L_s, L_m) = -\frac{3}{2} \left( \frac{1}{1 + e} \right)^{3/2} \times \)

\[ \Phi \left( \frac{(\frac{\mu}{\alpha} L_s)^2}{e} \right) - \Phi \left( \frac{(\frac{\mu}{\alpha} L_m)^2}{e} \right) \]

\[ \text{if } h < -(s + \frac{\mu}{\alpha \sigma^2}) \]

\[ \Phi \left( \frac{(\frac{\mu}{\alpha} L_s)^2}{e} \right) + \Phi \left( \frac{(\frac{\mu}{\alpha} L_m)^2}{e} \right) \]

\[ \text{if } -(s + \frac{\mu}{\alpha \sigma^2}) \leq h \leq m - \frac{\mu}{\alpha \sigma^2} \]

\[ \Phi \left( \frac{(\frac{\mu}{\alpha} L_s)^2}{e} \right) - \Phi \left( \frac{(\frac{\mu}{\alpha} L_m)^2}{e} \right) \]

\[ \text{if } h > m - \frac{\mu}{\alpha \sigma^2}, \]

is negative for all contract \((\alpha, h)\).

Notice that functions \( g(e, L_s, L_m) \) and \( g_e(e, L_s, L_m) \) are symmetric with respect to \( h \) around \( h^* = \frac{m-s}{2} - \frac{\mu}{\alpha \sigma^2} \), the center of the interval \([- (s + \frac{\mu}{\alpha \sigma^2}), m - \frac{\mu}{\alpha \sigma^2}]\). To see this, let \( \delta \) represent the deviation in the benchmark portfolio’s percentage invested in the risky asset above \( \delta > 0 \) or below \( \delta < 0 \) the reference value \( h^* \). It can be shown that \( L_s(h^* + \delta) = L_m(h^* - \delta) \) for all \( \delta \in \mathbb{R} \). Replacing the later equality in the functions \( g \) and \( g_e \), the symmetry is proved.

We call \( e_{TB} \) the third best effort that maximizes the constrained manager’s expected utility function in Proposition 1:

\[ e_{TB} = \arg \max_e - (1/2) \exp(- (1/2) \mu^2/\sigma^2 - aF + V(D, e)) \times g(e, L_s, L_m). \quad (10) \]

From (10) it is obvious that, unlike in the unconstrained scenario, the manager’s optimal effort depends on \( h \) (through \( L_s \) and \( L_m \)). We want to study how the third best effort changes with \( h \). In particular, does there exist an effort-maximizing benchmark?

The following proposition presents general conditions on the effort disutility function and the range of the benchmark parameter \( h \) for which there exists a well behaved effort function,
that is, a function that yields, for each benchmark portfolio $h$, the utility maximizing third best effort (10). More importantly, the same conditions are shown to be sufficient for the existence of a benchmark portfolio $h^*$ that elicits the highest effort from the manager. The value of $h^*$ is explicitly derived as a function of the manager’s portfolio constraints on short selling, $s$, and margin purchase, $m$; her relative risk aversion, $ao$; and the market portfolio moments, $\mu$ and $\sigma^2$. The fund’s Information Ratio (relative performance per unit of tracking error volatility) increases with effort. Given a contract $(F, \alpha)$, the benchmark $h^*$ is shown to yield the highest Information Ratio for the constrained fund.

**Proposition 2** Assume (S2)-(S4) hold. For all $h \in [-(s + \frac{\mu}{ao\sigma^2}), m - \frac{\mu}{ao\sigma^2}]$ there exists a unique function $e(h)$, continuous and differentiable, such that $e(h) = e_{TB}$. Let $h^* = \frac{m-s}{2} - \frac{\mu}{ao\sigma^2}$. Then, $e(h^*) > e(h)$ for all $h \neq h^* \in [-(s + \frac{\mu}{ao\sigma^2}), m - \frac{\mu}{ao\sigma^2}]$.

**Corollary 2** Assume (S2)-(S4) hold. Provided it exists, the effort function $e(h)$ is increasing in $h$ for all $h < -(s + \frac{\mu}{ao\sigma^2})$ and decreasing in $h$ for all $h > m - \frac{\mu}{ao\sigma^2}$. Moreover, the effort function is symmetric in $h$ around $h^*$, i.e., $e(h^* + \delta) = e(h^* - \delta)$ for all $\delta \in \Re$.

From Proposition 2 and Corollary 2, it is clear that the manager’s effort function attains a global maximum at $h^* = \frac{m-s}{2} - \frac{\mu}{ao\sigma^2}$. The intuition for this result is as follows: on the one hand, increasing benchmarking (i.e., higher $h$) lowers the likelihood of hitting the short selling constraint; on the other hand, it increases the probability of hitting the margin purchase constraint. The effect of decreasing benchmarking (i.e., lower $h$) is just symmetric. The trade-off of these two opposite effects yields the effort-maximizing value of the benchmark composition, $h^*$. In other words, the benchmark portfolio $h^*$ makes the manager, in expected terms, indifferent between hitting either constraint (short selling and margin purchase).

This intuition is illustrated in Figure 2. Assume first that the investor himself has to take the portfolio decision. The investor is constrained. For instance, $0 \leq \theta \leq 1$ (zero leverage). If the benchmark coincides with the risk free asset ($h = 0$), the investor will chose the tangent portfolio on the “absolute” capital market line that maximizes his expected utility. Notice that the slope of the capital market line coincides with the market Sharpe ratio, $\mu/\sigma$. His preferences are represented by the indifference curve $U(\theta)$. In the example, his optimal portfolio holds less than 50% in the market. If the investor were given a benchmark $h^* > 0$ then he will choose a tangent portfolio $\hat{\theta} = \theta - h^*$ in the “relative” capital market line that trades off excess expected return $\hat{\theta}\mu$ against tracking error standard deviation, $\hat{\theta}\sigma$. Notice that given the portfolio constraints, for $h = h^*$, the investor’s optimal unconditional portfolio is equidistant from either boundary. The investor, by assumption, has neither private information, nor the technology to acquire it. Therefore, the only implication of benchmarking would be a scaling of the optimal portfolio up to $\theta = 0.5$. The Information Ratio (excess return over the benchmark relative to the portfolio tracking error’s volatility) does not change and remains equal to the Sharpe Ratio.\[10\]

\[10\] For a given signal $y$, the Information Ratio is defined as $IR(y) = \frac{E(\hat{\theta}(y)|y)}{\sigma(\hat{\theta}(y)|y)}$. The unconditional Information Ratio will be $IR(\epsilon) = \int IR(y)dF(y) = \int E(\hat{\theta}(y)|y)dF(y) > \frac{\epsilon}{2}\sqrt{1+\epsilon}$, with $F(\cdot)$ the normal distribution function for the signal $y$. Figure 3 shows the Information Ratio as a function of the signal $y$ and given effort $e$. Notice that, when $e = 0$, the Information Ratio coincides with the Sharpe Ratio for all signal $y$. When $e$ increases, the slope
Assume now that the decision is taken by the manager who, in principle, has the ability to increase the precision of her private signal by putting some effort. The manager’s effort choice maximizes her unconditional expected utility before receiving the signal. The benchmark composition $h^*$ allows, ex-ante, more extreme signals to be implemented, increasing effort’s marginal utility and, ultimately, the manager’s effort choice. Therefore, the manager’s expected Information Ratio (i.e., averaging across all possible signals) changes relative to that of the the uninformed investors: it becomes higher than $\frac{\mu}{\sigma} \sqrt{1 + \epsilon}$. In fact, Proposition 2 shows that the benchmark $h^*$ maximizes effort expenditure and, therefore, the Information Ratio. Assuming, for simplicity, that the manager and the investor have the same risk aversion, the manager’s optimal portfolio would be greater than 50%.11

Proposition 2 shows that, under portfolio constraints, by choosing the appropriate benchmark, effort expenditure can be maximized. The immediate question is whether this benchmark choice may lead to an effort expenditure greater than the unconstrained, second best effort. The following proposition shows that, for any given contract and any portfolio composition, the effort choice for the constrained manager is smaller than for the unconstrained manager.

**Corollary 3** For any given contract $(F, \alpha, h)$ and finite manager’s risk aversion, $\alpha$, the constrained manager’s third best effort $e_{TB} < e_{SB}$. In the limit, when the portfolio constraints vanish, the third best effort and the second best effort coincide.

In other words, the model predicts that, other things equal, unconstrained managers will outperform constrained managers regardless of the composition of the benchmark used in the relative performance compensation of the latter.

We conclude this section by studying two special cases of the more general constrained problem. As illustrated in the examples in Section 3.2, when the manager is only short selling constrained (i.e., unlimited margin purchases), increasing the benchmark investment in the risky asset, $h$, gives the manager more incentives to exert greater effort. In the case of unlimited short selling and constrained margin purchases, the result is symmetric: effort decreases with $h$. In either case, there is no effort maximizing benchmark composition. The following corollary summarizes these findings.

**Corollary 4** When the manager can purchase at margin with no limit but faces a short selling bound, the effort function is monotonically increasing in $h$. Symmetrically, when the manager can sell short with no restriction but faces limited margin purchase, the effort function is monotonically decreasing with $h$.

increases in absolute value, making the Information Ratio higher for all signal $y$. As $\epsilon \to \infty$, in the limit, the Information Ratio also tends to infinity. For $y = \mu$, the Information Ratio becomes $\frac{\mu}{\sigma} \sqrt{1 + \epsilon}$. Averaging across $y$, the expected Information Ratio is higher than $\frac{\mu}{\sigma} \sqrt{1 + \epsilon}$ since for all $y < -\frac{\mu}{\epsilon}$, the Information Ratio “bounces back”: the manager would short the risky asset.

11As shown in Figure 3, the Information Ratio increases with the manager’s effort for all signal $y$. Averaging across signals, therefore, we obtain that the manager’s expected Information Ratio (i.e., the slope in Figure 2) increases with effort and given the contract $(F, \alpha)$ reaches a maximum at $h^*$. Notice that, at this point, we are not solving the investor’s optimal contract. Hence, no conclusion is drawn on $\alpha$. For an easier interpretation, assume $\alpha = 1$, such that the manager receives all the relative performance surplus.
5 The principal’s problem

The investor’s optimal contract \((F, \alpha, h)\) maximizes his expected utility subject to the manager’s incentive compatibility and participation constraints. For simplicity, and without loss of generality, we normalize the manager’s reservation value to \(-\exp(-(1/2)\mu^2/\sigma^2)\). For a given contract \((F, \alpha, h)\), the manager’s (conditional) wealth is given as a percentage, equation (2), of the fund’s net asset value.

The constrained manager, after accepting the contract, decides how much effort to exert. Subsequently, she receives the signal \(y\) and invests a proportion \(\theta(y)\) as in (6) in the risky asset. Let \(t(\alpha) = \frac{b(1-a)}{a}\) and \(T(\alpha) = (2-t(\alpha))t(\alpha)\). The investor’s expected utility is introduced in the following proposition.

**Proposition 3** Let \(a - eb > 0\) and \(a\alpha + eb (1 - \alpha(2 - t(\alpha))) > 0\). Given the portfolio constraints \(s \geq 0\) and \(m \geq 1\), the expected utility of the risk-averse investor is \(EU_b(\varphi_b(e)) = -\exp(b(F - R) - (1/2)\mu^2/\sigma^2) \times f(e, L_s, L_m)\) with the function \(f(e, L_s, L_m)\) defined in the Appendix.

The investor must choose the optimal linear contract, which includes the optimal fixed fee and the incentive fee, \(F\) and \(\alpha\), respectively, and the optimal benchmark, \(h\), subject to the participation constraint \(-\exp(-(1/2)\mu^2/\sigma^2 - aF + V(a, e)) \times g(e, L_s, L_m) \geq -\exp(-(1/2)\mu^2/\sigma^2)\). Clearly, neither effort nor \(h\) or \(\alpha\) are a function of \(F\). This, along with the fact that the left-hand side is increasing in \(F\) and the investor’s utility is decreasing in \(F\), implies that under the optimal contract, the participation constraint is binding. So, the investor’s expected utility can be expressed as a function of the contract \((\alpha, h)\), and the manager’s level of effort, \(e\):

\[
EU_b(\varphi_b(e)|\alpha, h) = -\exp(-bR - (1/2)\mu^2/\sigma^2 + (b/a)V(D, e)) \times g(e, L_s, L_m)^{b/a} f(e, L_s, L_m). \tag{11}
\]

We want to study how the portfolio constraints and the presence of moral hazard affect the investor’s optimal contract. We distinguish four cases depending on whether the manager’s effort is publicly observable or not (moral hazard) and whether the manager is constrained or unconstrained in her portfolio choice.

Assume first that the manager’s portfolio is unconstrained. If the manager’s effort decision is observable, the investor maximizes his expected utility with respect to \(\alpha, h\), and effort. We call this the first best scenario. We show then that the optimal contract is given by the first best incentive fee, \(\alpha_{FB} = b/(a+b)\), and zero benchmarking, \(h = 0\). The function \(f(e, L_s, L_m)\) becomes \(g(e)\). The investor chooses the first best effort level, \(e_{FB}\), that solves

\[
\max_e EU_b(\varphi_b(e)|\alpha_{FB}, 0) = -\exp(-bR - (1/2)(\mu/\sigma)^2 + (b/a)V(D, e)) g(e)^{\frac{a+b}{a}}.
\]

This results in the first order condition:

\[
V_e(D, e_{FB}) = \frac{1 + a/b}{2(1 + e_{FB})} = \frac{1/\alpha_{FB}}{2(1 + e_{FB})}.
\]

Notice that the higher the manager’s risk aversion (relative to the investor’s risk aversion), \(a/b\),
the lower the optimal incentive fee, $\alpha_{FB}$, and, consequently, the higher the investor’s participation in the fund’s return, $1 - \alpha_{FB}$. Hence, the investor becomes more interested in the manager’s signal precision: the marginal utility of effort increases and so does $e_{FB}$.\textsuperscript{12}

In the case when the manager’s effort decision is not observable, the investor’s problem consists in finding the optimal split that maximizes (11) subject to the manager’s optimal effort condition. Assume first that there exist no portfolio constraints. We call this scenario the second best. As shown in Section 3.1, the manager’s second best effort, $e_{SB}$, is independent of $\alpha$ and $h$. This result is consistent with Stoughton (1993) and Admati and Pfeiderer (1997). The investor will choose the same contract as in the first best case: $(\alpha_{FB}, 0)$. The second best effort satisfies the optimality condition (5):

$$V_e(D, e_{SB}) = \frac{1}{2(1 + e_{SB})}.$$ 

Comparing the latter two conditions, it is obvious that $e_{FB} > e_{SB}$ for all $a/b > 0$. That is, the second best effort coincides with the first best effort, the investor would choose himself in the limit when $b \to \infty$ (or, $a \to 0$) and, consequently, $\alpha_{FB} \to 1$. This would be equivalent to a swap contract between the manager (who takes all portfolio risk) and the investor (who gets, in exchange, a fixed rent, $F < 0$, from the manager). Notice that the manager’s marginal utility of effort is, in the second best case, independent of $a/b$ and $h$. Moreover, the cost (in terms of effort expenditure) of moral hazard increases with $a/b$: the investor would want to increase the manager’s effort but the contract fails to induce it.

We turn now to the case in which the manager’s portfolio is constrained. Assume first that the manager’s effort is observable. We show that the contract $(\alpha_{FB}, 0)$ is still optimal. The function $f(e, L_s, L_m)$ becomes $g(e, L_s(0), L_m(0))$. In this constrained first best scenario, the investor chooses the constrained first best effort level, $e^c_{FB}$, that maximizes $EU_b(\varphi_b(e)|\alpha_{FB}, 0) = -\exp(-bR - (1/2)(\mu/\sigma)^2 + (b/a)V(D, e))g(e, L_s(0), L_m(0))^{\frac{a+b}{b}}$

$$V_e(D, e^c_{FB}) = (1 + a/b) \frac{g(e^c_{FB}, L_s(0), L_m(0))}{g(e, L_s(0), L_m(0))}.$$ 

Notice that, as expected, portfolio constraints decrease the optimal effort choice: $e_{FB} > e^c_{FB}$.

Assume now that the manager’s effort is not observable. We call this scenario the third best. The manager’s third best effort satisfies (10). Section 3.2 shows that effort is increasing in $\alpha$ and, given $\alpha$, reaches an absolute maximum at $h^*$. We show that the contract $(\alpha_{FB}, 0)$ is no longer optimal. These results are presented in the following proposition.

**Proposition 4** Absent any portfolio constraint, the contract $(\alpha_{FB}, 0)$ is optimal, both for the public information case as well as under moral hazard.

Under portfolio constraints and no moral hazard, the contract $(\alpha_{FB}, 0)$ is still optimal. When the effort decision is not observable by the investor and hence there exists moral hazard, the contract $(\alpha_{FB}, 0)$ is suboptimal.

\textsuperscript{12}Here we are assuming that $D$ is independent of $a$. In case effort’s marginal disutility were increasing in $a$, the net effect on the first best effort would be unclear.
The implication of this proposition is that, to justify a benchmark different from the risk-free asset (or, in its absence, the minimum variance portfolio), both moral hazard and portfolio constraints must coexist. The following table summarizes the four possible scenarios and the optimal contract \((\alpha, h)\) in each of them:

<table>
<thead>
<tr>
<th>Effort observable</th>
<th>Effort unobservable</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unconstrained portfolio</strong></td>
<td><strong>Constrained portfolio</strong></td>
</tr>
<tr>
<td>FIRST BEST (FB)</td>
<td>CONSTRASTED FB (CFB)</td>
</tr>
<tr>
<td>((\alpha_{FB}, 0))</td>
<td>((\alpha_{FB}, 0))</td>
</tr>
<tr>
<td>SECOND BEST (SB)</td>
<td>THIRD BEST (TB)</td>
</tr>
<tr>
<td>((\alpha_{FB}, 0))</td>
<td>((\alpha_{TB}, h_{TB}))</td>
</tr>
</tbody>
</table>

We are interested in studying the optimal contract in the third best scenario, \((\alpha_{TB}, h_{TB})\). In spite of the simplifications, we cannot solve analytically for the general optimal contract under moral hazard and portfolio constraints. In the next section, we present a numerical solution to the problem.

6 A numerical solution of the third best contract

We propose the function \(V(D, e) = \frac{D}{2} e^2\) with disutility parameter \(D = 1\). Throughout the numerical analysis, we take the market excess return \(\mu = 6\%\) and the market volatility \(\sigma = 18\%\), both on an annual basis. The principal’s absolute risk aversion is \(b = 4\). The manager’s absolute risk aversion parameter takes values \(a = \{4, 8, 20\}\). We consider different degrees of portfolio constraints: \(s = 0\) and \(m = 1\) is the zero leverage base-case. We then allow for short selling \((s = 1\) and \(s = 2\)) and margin purchases \((m = 2\) and \(m = 3\)). For each combination \((a, b)\) and \((m, s)\), we calculate the manager’s effort and the investor’s expected utility \((11)\) for a grid of values for \(\alpha\) and \(h\) around the first best contract \((\alpha_{FB}(a, b), 0)\). \(\alpha\) changes from \(70\% \times \alpha_{FB}(a, b)\) to \(130\% \times \alpha_{FB}(a, b)\), at intervals of length \(5\%\). Likewise, \(h\) changes from \(-30\%\) to \(30\%\) at intervals of length \(5\%\). In the absence of moral hazard, for each contract \((\alpha, h)\), the manager puts the constrained first best effort that maximizes the investor’s expected utility in \((11)\). Under moral hazard, for each contract \((\alpha, h)\), the manager puts the third best effort in \((10)\).

Figure 4 introduces the base case under total constraints: \(m = 1\) and \(s = 0\) and risk aversion coefficients \(a = 4\) and \(a = 8\). Two scenarios are considered: Panel A presents the optimal contract in the absence of moral hazard (the manager’s effort decision is publicly observable), i.e., the constrained first best scenario; Panel B represents the optimal contract under moral hazard, i.e., the third best scenario. The investor’s expected utility is concave in \(\alpha\) and \(h\). In Panel A, we observe that, as predicted in proposition 4, in the absence of moral hazard, the investor’s maximum expected utility is attained at the first best contract \((\alpha_{FB}, 0)\) with zero benchmarking.

The figures in Panel B confirm the prediction in Proposition 4: the first best contract is no longer optimal in the presence of moral hazard. In concrete, the incentive fee increases from
50% to 60% for \( a = 4 \); from 33% to 45% for \( a = 8 \). The benchmark becomes more risky: the percentage invested in the market portfolio rises from zero to 10% for \( a = 4 \) and to 15% for \( a = 8 \).

Table 1 summarizes the optimal contracts \((\alpha_{TB}, h_{TB})\) under moral hazard and different levels of portfolio constraints. Let us first concentrate on the totally constrained scenario, that is, \( m = 1 \) and \( s = 0 \). Table 1 shows that, when the manager’s risk aversion \( a \) increases, the optimal incentive fee decreases (as it would be expected) although it is always higher than in the first base case. Concretely, \( \alpha_{TB} \) is, respectively, 20% \((a = 4)\), 35% \((a = 8)\) and 65% \((a = 20)\) higher than the first best \( \alpha_{FB} \). Therefore, relative to the unconstrained case, the percentage increase in the contract’s optimal incentive fee grows with the manager’s risk aversion. Looking now at the benchmark composition, we observe that \( h \) increases from zero to 10% \((a = 4)\), 15% \((a = 8)\), and 20% \((a = 20)\). These results suggest that the investor has more incentives to alter the first best contract under portfolio constraints (by increasing both the incentive fee, \( \alpha \), and \( h \)) when the manager is more risk averse. This is in agreement with the analytical results in Section 5. We showed then that the investor’s utility loss in the unconstrained, second best scenario increased with the manager’s risk aversion relative to the first best case. The argument is the following: the investor becomes more focussed on the manager’s ability (effort) as he retains a larger proportion of the portfolio output \((\alpha_{FB} \text{ decreases with } a/b)\). In the absence of portfolio constraints, the second best contract (the same as the first best) fails to induce higher effort on the manager. However, in the presence of portfolio constraints, both the incentive fee and the benchmark composition become relevant in reducing the inefficiency caused by the moral hazard problem and the portfolio constraints. Additionally, their efficiency increases with the manager’s risk aversion.

Looking now at the effect of relaxing the portfolio constraints we observe the following. On one hand, holding the short selling limit, \( s \), constant and increasing \( m \) results in a lower incentive fee, \( \alpha_{TB} \), and a higher optimal benchmark, \( h_{TB} \). On the other hand, holding the margin purchase constraint constant and increasing the short selling constraint implies both a lower \( \alpha_{TB} \) and \( h_{TB} \). The effect on the benchmark coincides with the predictions of Corollary 4. Notice that when \( m = 3 \), \( s = 2 \) and \( a = 20 \), the optimal third best contract coincides with the unconstrained, second best contract \((16,0)\).

We are interested now in analyzing the implications of the optimal contract on the manager’s effort choice and the investor’s welfare. As a measure of the latter we use the investor’s certainty equivalent wealth (CEW).\(^1\) To estimate the effect of the optimal contract on the manager’s effort expenditure, we can look at Table 2. Panel A in this table shows the third best effort for each optimal contract in Table 1. Recall that effort can be interpreted as the percentage net increase in precision induced by the manager’s private information. Concentrating on the constrained setting \((m = 1 \text{ and } s = 0)\), we observe that the manager’s effort increases with her risk aversion.

\(^1\)Given the investor’s utility function, \( U_b(W) = -\exp(-bW) \), the certainty equivalent wealth of the expected utility \( u \) is given by the inverse of this function, \( C(u) = -\ln(-u)/b \). Clearly, for any two values of the investor’s expected utility, \( u_1 \) and \( u_2 \), \( u_1 > u_2 \) if and only if \( C(u_1) > C(u_2) \). In concrete, given equation (11), for a given expected utility value \( u = -\exp(-bR - (1/2)b^2)a^2 + (b/a)V(D,e)) \times g(e,L_e,L_m) \), then \( C(u) = R = (1/2b)\sigma^2 / \sigma^2 - (1/a)(V(D,e) + \ln g(e,L_e,L_m)) - (1/b)\ln f(e,L_e,L_m) \). We call \( CEW(u) = C(u) - R \), the excess risk-free return (above the bond’s return, \( R \)) that leaves the investor indifferent.
from 16% \((a = 4)\) to 26% \((a = 20)\). As a benchmark, the table also reports the unconstrained effort levels without (first best) and with information asymmetry and moral hazard (second best): the first best effort is 61.8% \((a = 4)\), 82.29% \((a = 8)\), 130.28% \((a = 20)\), respectively. The second best effort is 36.63%, independently of \(a\). Therefore, even when the manager is totally constrained, when properly compensated, her effort exertion (and the corresponding timing ability) can be quite substantial and increasing in the manager’s risk aversion.

Panel B shows, for every contract and portfolio constraint, the percentage change in the third best optimal effort in Panel A with respect to the effort the manager would exert if compensated with the suboptimal first best contract, \((\alpha_{FB}, 0)\). The inefficiency caused by a suboptimal compensation, both in terms of the incentive fee and the benchmark composition can be very substantial: the manager’s effort (hence, timing ability) would be reduced by almost 30% in the case of \(a = 20\). When portfolio constraints are relaxed, the loss in effort expenditure from the suboptimal contract decreases.

In order to disentangle the effect of \(h\) and \(\alpha\) on effort expenditure, Figure 5 presents the percentage variation in effort, relative to the third best, when we change the benchmark composition \(h\) holding constant the optimal third best incentive fee, \(\alpha_{TB}\), in Table 2. For \(a = 4\), the investor could increase the manager’s effort by 5% reducing the benchmark’s investment in the risky asset from \(h_{TB} = 10\%\) to \(h^* = -27.16\%\).\(^{14}\) This would be, however, suboptimal since the investor’s certainty equivalent wealth would decrease from 2.67% down to 1.97%. When \(a = 20\), the third best optimal benchmark, \(h_{TB} = 20\%\), is very close to \(h^* = 16.33\%\). By moving down to \(h^*\), the manager’s effort barely increases by 0.1%.

This is an important implication of our model: depending on the manager’s risk aversion, the investor may optimally forgo higher effort inducement (timing ability) on the manager by moving away from the highest effort benchmark \(h^*\) and therefore, the highest Information Ratio. The intuition behind this result lies in the balance between the incentives for effort expenditure (which increases the investor’s expected utility) and the distortion that benchmarking introduces by leading the manager to a suboptimal risk-return trade-off (Roll’s critique to benchmarking). When the manager’s risk aversion is relatively small (in our example, for \(a = 4\)), the investor’s part in the portfolio’s return \((\alpha)\) decreases and so does his marginal utility from the manager’s effort. The investor’s concern about the manager’s risk taking dominates the role of the benchmark in providing managerial effort incentives. Only when the manager’s risk aversion is high enough (in our example, for \(a = 20\)) the marginal utility gain from extra effort expenditure compensates the investor for the higher total risk exposure induced by a larger \(h_{TB}\).

We turn now into the effects of changing the contract on the investor’s utility (represented by his CEW). Table 3 presents the CEW loss that would result for the investor from offering the manager a suboptimal first best contract \((\alpha_{FB}, 0)\). The figures could be interpreted as the annual excess risk-free return (above the bond’s return, \(R\)) that would compensate the investor for the loss in expected utility from the suboptimal first best contract. Notice that the loss is higher the more constrained the manager is and the larger is her risk aversion, \(a\). In fact, when \(s = 2\) and \(m = 3\) the optimal third best contract for a manager with risk aversion coefficient

\(^{14}\)Recall that \(h^*\) is the effort maximizing benchmark composition in the manager’s partial equilibrium problem, Section 4.
\( a = 20 \) is, according to Table 1, \((16,0)\), the same as the first best contract. Consequently, the CEW loss is zero. In contrast, when the same manager is fully constrained, the CEW loss increases above 4.5%.

We calculate the investor’s CEW loss relative to the optimal third best contract when we give the manager the optimal third best incentive fee, \( \alpha_{FB} \) and change the benchmark composition \( h \). Figure 6 illustrates this loss for three different values of \( a \), the manager’s risk aversion. The CEW loss increases with the manager’s risk aversion and is symmetric around \( h_{FB} \); that is, overbenchmarking as well as underbenchmarking decrease the investor’s welfare. For \( a = 20 \), the percentages of CEW loss can be higher than 4% when \( h \) goes beyond 40%.

Summarizing the results from this section, the numerical solution of the investor’s optimal contract and the sensitivity analysis performed on the benchmark choice, we conclude that:

1. When the manager is fully constrained (neither selling short nor purchasing at margin are allowed), choosing the optimal benchmark has quantitatively important implications for the manager’s timing ability and the investor’s welfare. It is also optimal to increase (relative to the unconstrained case) the manager’s incentive fee. The inefficiency in terms of effort expenditure (for the manager) and welfare (for the investor) from a suboptimal contract increase with the manager’s risk aversion.

2. When the manager is constrained, it is not necessarily optimal for the investor to maximize the portfolio’s Information Ratio. The deviation from the potentially highest Information Ratio decreases with the manager’s risk aversion.

3. When portfolio constraints are relaxed, the optimal benchmark changes accordingly: it increases (decreases) when, ceteris paribus, the margin purchase (short selling) constraint is relaxed. In any case, holding the manager’s risk aversion constant, the optimal incentive fee decreases towards the first best incentive fee. For the investor, when either constraint is relaxed (holding the other), the efficiency cost of choosing the wrong benchmark decreases although it remains sizeable for the case of risk averse managers.

7 Introducing stock-picking ability

So far in this paper, for the sake of simplicity, we have assumed that there exists one single risky asset and that the manager can only show timing ability.

The evidence, however, shows (Elton, Gruber and Blake, 2003) that a common strategy among mutual funds is to attempt to outperform a benchmark by taking exposure to additional systematic risk factors that are priced. Moreover, managers may have both timing and selectivity or stock-picking ability. In this section, we generalize the model by including a second risky asset and endowing the manager with stock-picking ability.

Assume now that there are two risky assets and a risk-less bond. Let the risky assets excess return be denoted by \((x,z)\). They are independent and normally distributed:

\[
\begin{pmatrix} x \\ z \end{pmatrix} \sim \mathcal{N}\left( \begin{pmatrix} \mu_x \\ \mu_z \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_z^2 \end{pmatrix} \right).
\]
We follow the “portfolio model” of timing and selectivity in Admati et al. (1986): $x$ represents the excess return on a timing portfolio; $z$ is the excess return on another security or portfolio with unconditional risk premium $\mu_z$ and volatility $\sigma^2_z$. We call $z$ the selectivity portfolio. We assume that these two portfolios are orthogonal. Portfolio $z$ would represent investment strategies like investing in small or value stocks (analogously, shorting large or growth stocks) which deliver unconditional positive risk premium.

The manager and the investor have the same preferences as in the simpler model. Likewise, the contract offered to the manager has the same structure and parameters, namely, $F$ and $\alpha$. After learning the contract, the manager puts some non-observable effort. Unlike in the previous setting, we now distinguish between two types of effort and, accordingly, two types of managerial skills. On the one side, the manager may have a timing ability, represented by $e_x \geq 0$, to shift between the risk-free asset and the timing portfolio. Independently, the manager may also develop a stock picking or selectivity skill represented by $e_z \geq 0$: by putting effort, the manager will learn, for instance, to select mispriced small (large) or value (growth) stocks. More concretely, assume for example a mutual fund with a growth objective; $x$ would represent the return on a growth stock portfolio. With zero timing effort/ability, we expect that the manager’s optimal portfolio would invest a percentage directly proportional to the risk-adjusted growth premium and inversely proportional to her risk aversion. If the manager has timing ability, according to the model in Section 3.2, the manager’s investment in the growth portfolio in $t - 1$ should increase with the portfolio’s realized performance in $t$. Besides investing in the growth portfolio, the manager could invest in $z$ representing, for instance, the return on a small-minus-big capitalization (SMB) strategy. The investment in this strategy depends on the unconditional moments of the SMB portfolio and the manager’s risk aversion. If the manager possesses some selectivity ability, she could use it to deliver excess return above the risk-adjusted return for $z$. This would be captured by the Jensen’s alpha.

If the manager accepts the contract, she decides both levels of effort in acquiring private (non observable) information that materializes in two independent signals:

$$y_x = x + \frac{\sigma_x}{\sqrt{e_x}} \epsilon_x,$$
$$y_z = z + \frac{\sigma_z}{\sqrt{e_z}} \epsilon_z.$$

Noise terms follow a standard normal distribution. Following the traditional approach in the literature, we assume that selectivity information is independent of market timing information. This implies that both noise terms are orthogonal to $x$ and $z$ and uncorrelated. In other words, by observing the market timing private signal, the manager learns nothing about stock picking and vice-versa. Assumption (S1) is reformulated as:

**Assumption (S1)** $E(i \epsilon_i) = E(\epsilon_x \epsilon_z) = 0$ for $i = \{x, z\}$.

The greater the effort, the higher the corresponding’s signal’s precision. Conditional on the manager’s effort and the signal realization, the timing portfolio’s excess return is normally distributed with mean excess return $E(x|y_x) = \frac{\mu_x + e_x y_x}{1 + e_x}$ and conditional precision $\text{Var}^{-1}(x|y_x) = \frac{1}{\sigma^2_x}(1 + e_x)$. Analogously, the conditional excess return on the individual security will be normally distributed.
with mean excess return $E(z|y_z) = \mu_x + e_y y_z$ and conditional precision $\text{Var}^{-1}(z|y_z) = \frac{1}{\sigma_z^2}(1 + e_z)$. Given our assumptions, the conditional returns are uncorrelated.

Effort is costly. We redefine the effort disutility function $V(D, e)$ to accommodate the manager’s timing and selectivity abilities. Hence, $D = (D_x, D_z)$ and $e = (e_x, e_z)$ with $D_i$ the corresponding effort disutility parameter for timing ($i = x$) and selectivity ($i = z$) effort. Therefore, assumptions (S2) and (S3) still hold and are conveniently scaled to this new two-dimension setting. We also assume that the cross second derivative of effort disutility is zero: $V(D, e)e_x e_z = 0$. This implies that market timing and selectivity effort disutility are independent.\(^{15}\)

Assumption (S4) is rewritten as

**Assumption (S4)** $\frac{V_{e_i e_i}(D, e)}{V_{e_i}(D, e)} > \frac{1}{1+e_i}$ for $i = \{x, z\}$.

We now revisit the unconstrained portfolio choice in Section 3.1. Based on the conditional moments, the manager decides what percentage of the fund’s net asset value to invest in the timing portfolio, $\theta_x(y_x)$ and what percentage to invest in the individual security $\theta_z(y_z)$; the remaining, $1 - \theta_x(y_x) - \theta_z(y_z)$ is invested in the risk-free bond. Let $\theta = (\theta_x, \theta_z)'$ and $y = (y_x, y_z)'$. Therefore, the portfolio’s return will be $R_p = R + (x, z)\theta$.

The benchmark is defined as a combination of the timing portfolio and the risk-free bond with return $R_b = R + h x$. The portfolio’s net return over the benchmark is given by $R_p - R_b = (x, z)\theta$ with $\theta = (\theta_x - h, \theta_z)'$.

Following the same procedure as in Section 3.1, we obtain the unconstrained conditional portfolio $\bar{\theta}(y) = \frac{1}{a_0} \left( \frac{\mu_y e_x y_x}{\sigma_y^2}, \frac{\mu_y e_z y_z}{\sigma_y^2} \right)'$ and the manager’s unconditional expected utility:

$$EU(\varphi_u(e)) = -\exp\left(-\frac{1}{2}\left(\frac{\mu_y^2}{\sigma_y^2} - \frac{1}{2}\right)\left(\frac{\mu_y^2}{\sigma_y^2} - aF + V(D, e)\right)g(e_x)g(e_z), \right) \quad (12)$$

with $g(e_i) = \left(\frac{1}{1+e_i}\right)^{1/2}$. Equation (5) shows the the optimal second best effort $e_i$ with $i = \{x, z\}$. Timing and selectivity effort choices are independent of the contract and the benchmark parameter $h$. They only depend on the effort disutility parameters in $D$.

We now tackle the constrained portfolio choice. We assume for simplicity, and without loss of generality, that the leverage constraints, $s$ and $m$, are the same for both risky assets. The percentage invested in the timing portfolio, $\theta_x(y_x)$, will be a function of the timing signal $y_x$. The proportion invested in the selectivity asset, $\theta_z(y_z)$, will be a function of the selectivity signal $y_z$. $\theta_x(y_x)$ coincides with portfolio (6) in Section 3.2 with $y$, $e$ and $\sigma^2$ replaced by $y_x$, $e_x$ and $\sigma_x^2$, respectively.

The investment in the selectivity portfolio is be given by

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\(^{15}\)See Van Nieuwerburgh and Veldkamp (2008) and Kacperczyc, Van Nieuwerburgh, and Veldkamp (2009) for a model with strategic choice of learning and its implications for timing and selectivity skills.
Averaging across $y_x$ and $y_z$, we obtain the manager’s unconditional expected utility $EU(\varphi_b(e)) = \exp(-(-1/2)(\mu_x^2/\sigma_x^2) - (1/2)(\mu_z^2/\sigma_z^2) - aF + V(D, e)) g(e_x, L_s, L_m) g(e_z, s, m)$ with $g(e_x, L_s, L_m)$ as in Proposition 1 and $g(e_z, s, m) = (1/2) \times$

$$
\exp\left(\left(\frac{sa \alpha \sigma_z}{2}\right)^2\right) \left[1 - \Phi\left(\frac{1+e_x}{e_z} \left(\frac{sa \alpha \sigma_z}{e_z}\right)^2\right)\right] + \\
\left(\frac{1}{1+e_x}\right)^{1/2} \left[\Phi\left(\frac{sa \alpha \sigma_z}{e_x}\right) + \Phi\left(\frac{ma \alpha \sigma_z}{e_x}\right)\right] + \\
\exp\left(\left(\frac{ma \alpha \sigma_z}{2}\right)^2\right) \left[1 - \Phi\left(\frac{1+e_x}{e_x} \left(\frac{ma \alpha \sigma_z}{e_x}\right)^2\right)\right],
$$

independent of $h$.

From Corollary 1, $g_{e_x}(e_x, s, m) = -\frac{1}{4} \left(\frac{1}{1+e_x}\right)^{3/2} \left[\Phi\left(\frac{1+e_x}{e_x} \left(\frac{sa \alpha \sigma_z}{e_x}\right)^2\right) + \Phi\left(\frac{ma \alpha \sigma_z}{e_x}\right)\right] < 0$.

Given the contract $(F, \alpha, h)$, replacing $\sigma_z^2$ for $\sigma^2$, equation (10) yields the optimal timing third best effort $e_x$. Corollaries 2 through 4 hold independently of the selectivity effort $e_z$. Likewise, replacing $\sigma_z^2$ for $\sigma^2$, $sa \alpha \sigma_z$ for $\frac{p}{\alpha} L_s$, $ma \alpha \sigma_z$ for $\frac{p}{\alpha} L_m$ and $g(e, s, m)$ for $g(e, L_s, L_m)$, equation (10) yields the optimal selectivity third best effort, independent of $h$, the benchmark composition. Corollary 3 also holds for the third best selectivity effort. That is, both for timing and selectivity skills, the unbounded second best effort is greater than the constrained third best effort and they coincide in the limit, as portfolio limits vanish.

Let us turn now to the principal’s problem. Analogously to the reformulation of the manager’s expected utility function in the presence of selectivity ability, the investor’s expected utility function in Proposition 3 becomes $EU_b(\varphi_b(e)) = -\exp(b(F - R) - (1/2)\mu_x^2/\sigma_x^2 - (1/2)\mu_z^2/\sigma_z^2) \times f(e_x, L_s, L_m) f(e_z, s, m)$ with the function $f(e_x, L_s, L_m)$ as defined in the Appendix (with $e$ and $\sigma^2$ replaced by $e_x$ and $\sigma_x^2$, respectively) and function $f(e_z, s, m) = (1/2) \times$

$$
\exp\left(\left(\frac{(t(a) sa \alpha \sigma_z)}{2}\right)^2\right) \left[1 - \Phi\left(\frac{1+(t(a)e_x)}{1+e_z} \left(\frac{sa \alpha \sigma_z}{e_z}\right)^2\right)\right] + \\
\left(\frac{1}{1+(t(a)e_x)}\right)^{1/2} \left[\Phi\left(\frac{1+(t(a)e_x)}{1+e_x} \left(\frac{sa \alpha \sigma_z}{e_x}\right)^2\right) + \Phi\left(\frac{1+T(a)e_x}{1+e_x} \left(\frac{ma \alpha \sigma_z}{e_x}\right)^2\right)\right] + \\
\exp\left(\left(\frac{(t(a) ma \alpha \sigma_z)}{2}\right)^2\right) \left[1 - \Phi\left(\frac{1+(t(a)e_z)}{1+e_x} \left(\frac{ma \alpha \sigma_z}{e_x}\right)^2\right)\right].
$$

Notice that for the first best $\alpha_{FB} = b/(a + b)$, $g(e_z, s, m) = f(e_z, s, m)$.

After these definitions, the investor’s expected utility can be written as a function of $(\alpha, h)$.
like in equation (11):

$$EU_b(\varphi_b(c)|\alpha, h) = -\exp(-bR - (1/2)\mu_z^2/\sigma_z^2 - (1/2)\mu_z^2/\sigma_z^2 + (b/a)V(D, c)) \times$$

$$g(e_z, L_s, L_m)^{b/a} g(e_z, s, m)^{b/a} f(e_z, L_s, L_m) f(e_z, s, m).$$  \hspace{1cm} (15)$$

The Appendix shows that the results in Proposition 4 hold in the presence of selectivity information. Concretely, the contract \((\alpha_{FB}, 0)\) is shown to be suboptimal. Notice also that, given (15), since \(f(e_z, s, m)\) and \(g(e_z, s, m)\) are independent of \(h\), therefore, \(e_z\) will be independent of \(h\) as well. It is only a function of \(\alpha\) and the manager’s effort disutility. This implies that the numerical results with respect to \(h\) and its relation to the portfolio constraints in Section 6 will remain qualitatively unchanged.

8 Conclusions

This paper investigates the effort inducement incentives of (potentially benchmarked) linear incentive fee contracts. Incentives arise explicitly via the compensation of the manager. The investor has to decide simultaneously the incentive fee (the manager’s participation in the delegated portfolio’s return) and the benchmark composition.

The contribution of our paper to the literature on management compensation comes from the fact that we incorporate portfolio constraints in our model. These constraints are exogenous in our model and could be motivated by regulation or, as suggested by Almazan et al. (2004), as alternative monitoring mechanism in a broader equilibrium model.

Under portfolio constraints and moral hazard, our model derives a new set of predictions. The portfolio manager should be offered an incentive fee (performance related) higher than in the absence of portfolio constraints. This incentive fee should be benchmarked against a portfolio that combines the risky market portfolio and the riskless asset. When the benchmark design is endogenous, maximizing the Information Ratio may turn suboptimal for the fund investor: depending on the manager’s risk aversion (relative to the investor’s) the increase in the manager’s timing ability may not compensate for the excessive risk exposure. Only when the manager is sufficiently risk averse, maximizing the Information Ratio becomes optimal for the investor. These results are in contrast with the predictions from the unconstrained setting in Adamati and Pfleiderer (1997), where the risk-free asset was the optimal benchmark. When portfolio constraints are removed, the model predicts that the manager’s effort is unrelated to the incentive fee and the benchmark composition, a well-known result in the literature.

These results are consistent with the prevalence of absolute return (non-benchmarked) compensation schemes among hedge fund managers, arguably much less constrained than mutual fund managers. Golec (1992) and Elton, Gruber and Blake (2003) document that the number of mutual funds that explicitly use incentive fees is relatively small in comparison with the pervasive use of a “flat” fee (a fixed percentage of the fund’s net asset value). However, Elton, Gruber and Blake (2003) find that funds which use incentive fees have superior performance.
relative to those that do not. In their conclusions, they claim that “while at this time funds with incentive fees seem to offer superior performance relative to other actively managed funds, we don’t know whether this is true because of the motivation supplied by incentive fees or because skilled managers adopt incentive fees to advertise their skills to the public.” Our model shows that under portfolio constraints and moral hazard, portfolio managers who are offered a properly benchmarked incentive fee are more motivated than equally skilled managers whose compensation is not performance-linked.

References


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16 Agarwal, Daniel, and Naik (2009) find that even for hedge funds, the call-option-like incentive fee contract provides incentives to deliver superior performance. In particular, they find that funds with higher delta have better future performance.


Brennan, M. J. (1993), Agency and Asset Pricing, working paper, UCLA.


Appendix

Proof of Proposition 1

Replacing (6) in the manager’s utility function:

$$EU(\varphi_a(y)) = -\exp(-\alpha F + V(D,e)) \times$$

$$\exp\left((h+s)\alpha E(x|y) + (1/2)((h+s)\alpha)^2 \text{Var}(x|y)\right)$$

if $y < -\frac{\mu}{s} L_s$

$$\exp\left(-\frac{1}{2}E^2(x|y)/\text{Var}(x|y)\right)$$

otherwise

$$\exp\left(-(m-h)\alpha E(x|y) + (1/2)((m-h)\alpha)^2 \text{Var}(x|y)\right)$$

if $y > \frac{\mu}{s} L_m$.

Multiplying the previous expression by the density function of the signal variable, $y$, we obtain:

$$-\exp\left(-\frac{1}{2}(\mu^2/\sigma^2) - \alpha F + V(D,e)\right) \left(\frac{e}{1+e}\right)^{1/2} \frac{1}{\sqrt{2\pi}\sigma} \times$$

$$\exp\left(-\frac{1}{2}\frac{e}{1+e} \left(\frac{e}{\sigma} - \frac{\mu}{s} L_s\right)^2\right)$$

if $y < -\frac{\mu}{s} L_s$

$$\exp\left(-\frac{1}{2}e \left(\frac{e}{\sigma}\right)^2\right)$$

otherwise

$$\exp\left(-\frac{1}{2}\frac{e}{1+e} \left(\frac{e}{\sigma} + \frac{\mu}{s} L_m\right)^2\right)$$

if $y > \frac{\mu}{s} L_m$.

Replace $k = \frac{e}{1+e} \left(\frac{e}{\sigma} - \frac{\mu}{s} L_s\right)^2$ if $y < -\frac{\mu}{e} L_s$; $k = \frac{e}{1+e} \left(\frac{e}{\sigma} + \frac{\mu}{s} L_m\right)^2$ if $y > \frac{\mu}{e} L_m$, and $k = e \left(\frac{e}{\sigma}\right)^2$ otherwise. Integrating over $k$ and given the definition of $\Phi(\cdot)$, the unconditional utility function follows. QED

Proof of Corollary 1

By definition, $|L_m| > |L_s|$ for all $-\infty < h < -(s + \frac{\mu}{\alpha \sigma^2})$ such that $\Phi\left(\frac{\left(\frac{\mu}{s} L_s\right)}{e}\right) - \Phi\left(\frac{\left(\frac{\mu}{s} L_s\right)^2}{e}\right) > 0$; likewise $|L_s| > |L_m|$ for all $\infty > h > m - \frac{\mu}{\alpha \sigma^2}$ such that $\Phi\left(\frac{\left(\frac{\mu}{s} L_m\right)^2}{e}\right) - \Phi\left(\frac{\left(\frac{\mu}{s} L_m\right)^2}{e}\right) > 0$. QED

Proof of Proposition 2

Let us define $J(e, L_s, L_m) = V_e(D,e) \times g(e, L_s, L_m) + g_e(e, L_s, L_m)$. The function $J \in C^1$ for all $(e, h)$. The third best effort in (10) satisfies:
\[ \mathcal{J}(e_{TB}, L_s, L_m) = 0, \quad \text{(A1)} \]
\[ \mathcal{J}_e(e_{TB}, L_s, L_m) > 0. \quad \text{(A2)} \]

The implicit function theorem allows us to solve “locally” the equation; that is, for all \((\hat{e}, \hat{h})\) that satisfy (A1) and (A2), effort \(e\) can be expressed as a function of \(h\) in a neighborhood of \((\hat{e}, \hat{h})\).

More formally: for all \((\hat{e}, \hat{h})\) that satisfy (A1) and (A2) there exists a function \(e(h) \in \mathcal{C}^1\) and an open ball \(B(\hat{h})\), such that \(e(\hat{h}) = e_{TB}\) and \(\mathcal{J}(e(h), L_s, L_m) = 0\) for all \(h \in B(\hat{h})\).

Taking the derivative of \(\mathcal{J}(e_{TB}, L_s, L_m)\) with respect to \(h\):\(^{17}\)

\[ e_h(h) = -\mathcal{J}_h(e_{TB}, L_s, L_m) \times \mathcal{J}^{-1}_e(e_{TB}, L_s, L_m). \]

Taking the second derivative of (8) with respect to \(e\):

\[ g_{ee}(e, L_s, L_m) = \frac{1}{2} \left( \frac{1}{1+e} \right)^{3/2} \left\{ \frac{3}{2} \left[ \frac{1}{1+e} \right] \left[ \Phi \left( \frac{(\frac{\mu}{\sigma} L_s)^2}{e} \right) + \Phi \left( \frac{(\frac{\mu}{\sigma} L_m)^2}{e} \right) \right] + \frac{1}{e^2} \left[ \phi \left( \frac{(\frac{\mu}{\sigma} L_s)^2}{e} \right) \times \phi \left( \frac{(\frac{\mu}{\sigma} L_m)^2}{e} \right) \right] \right\} > 0. \]

Condition (A2) can be written as \(V_{ee}(D, e) > -\frac{\eta_2}{g}(e, L_s) \times V_e(D, e) - \frac{\eta_2}{g}(e, L_s)\). Then, (S4) implies (A2) for all \(h \in [-s + \frac{\mu}{\alpha\sigma^2}, m - \frac{\mu}{\alpha\sigma^2}].\)

The sign of \(e_h(h)\), therefore, depends on the sign of \(\mathcal{J}_h(e, L_s, L_m) = V_e(D, e) \times g_h(e, L_s, L_m) + g_{eh}(e, L_s, L_m)\).

From (S3), \(V_e(D, e) > 0\). From Corollary 1,

\[ g_{eh}(e, L_s, L_m) = -\left( \frac{1}{1+e} \right)^{3/2} e^{-1/2} \frac{a\alpha\sigma}{\sqrt{2\pi}} \left[ \exp \left( -\frac{(\frac{\mu}{\sigma} L_s)^2}{2e} \right) - \exp \left( -\frac{(\frac{\mu}{\sigma} L_m)^2}{2e} \right) \right] \quad \text{(A3)} \]

for all \(h \in \mathbb{R}\).

Let us define the gamma function \(\Gamma(u) = \int_0^\infty t^{u-1}\exp(-t)dt\) for \(u > 0\). The incomplete gamma function is given by \(\Gamma(u, v) = \int_v^\infty t^{u-1}\exp(-t)dt\) for \(v > 0\). From (8),

\[ g_h(e, L_s, L_m) = \frac{a\alpha\mu}{\sqrt{\pi}} \Gamma \left( \frac{1}{2}, \frac{1}{1+e} \right) \left( L_s \exp \left( \frac{(\frac{\mu}{\sigma} L_s)^2}{2} \right) - L_m \exp \left( \frac{(\frac{\mu}{\sigma} L_m)^2}{2} \right) \right) - \left( \frac{e}{1+e} \right)^{1/2} \frac{2a\alpha\sigma}{\sqrt{2\pi}} \left[ \exp \left( -\frac{(\frac{\mu}{\sigma} L_s)^2}{2e} \right) - \exp \left( -\frac{(\frac{\mu}{\sigma} L_m)^2}{2e} \right) \right]. \quad \text{(A4)} \]

\(^{17}\)The subscript \(h\) denotes first derivative with respect to \(h\). The subscript \(eh\) denotes cross derivative with respect to \(e\) and \(h\).
By definition, \( L_s(h^* + \delta) = L_m(h^* - \delta) \), for all \( \delta \in \mathbb{R} \). For all \( 0 < \delta < \frac{m + s}{2} \), \( L_s(h^* - \delta) < L_m(h^* - \delta) \) and \( L_s(h^* + \delta) > L_m(h^* + \delta) \). Let \( L_s^* = L_s(h^*) \) and \( L_m^* = L_m(h^*) \). For \( \delta = 0 \), \( L_s^* = L_m^* \). Therefore, \( e_h(h) > 0 \) for all \( s + \frac{\mu}{a\sigma^2} < h < h^* \) and \( e_h(h) < 0 \) for all \( h^* < h \leq m - \frac{\mu}{a\sigma^2} \); \( e_h(h^*) = 0 \). Since the function \( e(h) \) is continuous and differentiable, it follows that \( h^* \) is a local maximum in the interval \([- (s + \frac{\mu}{a\sigma^2}), m - \frac{\mu}{a\sigma^2}] \). \( Q.E.D. \)

**Proof of Corollary 2**

Let \( h < -(s + \frac{\mu}{a\sigma^2}) \). Then, \( L_s < 0 \) and \( L_m > 0 \) and \( |L_s| < |L_m| \). From (7),

\[
g_h(e, L_s, L_m) = a\alpha \mu L_s \exp \left( \frac{(\frac{\mu}{\sigma} L_s)^2}{2} \right) \left[ 1 + \Phi \left( \frac{1 + e}{\sigma} \left( \frac{\mu}{\sigma} L_s \right)^2 \right) \right] - a\alpha \mu L_m \exp \left( \frac{(\frac{\mu}{\sigma} L_m)^2}{2} \right) \left[ 1 - \Phi \left( \frac{1 + e}{\sigma} \left( \frac{\mu}{\sigma} L_m \right)^2 \right) \right] - \left( \frac{e}{1 + e} \right)^{1/2} \frac{2a\alpha \sigma}{\sqrt{2\pi}} \left[ \exp \left( \frac{-((\frac{\mu}{\sigma} L_s)^2)}{2e} \right) - \exp \left( \frac{-((\frac{\mu}{\sigma} L_m)^2)}{2e} \right) \right] < 0
\]

From (A3), \( g_h(e, L_s, L_m) < 0 \). Given (S3), it follows that \( e_h(h) > 0 \) for all \( h < -(s + \frac{\mu}{a\sigma^2}) \).

Let \( h > m - \frac{\mu}{a\sigma^2} \). Then, \( L_s > 0 \) and \( L_m < 0 \) and \( |L_s| > |L_m| \). From (9),

\[
g_h(e, L_s, L_m) = a\alpha \mu L_s \exp \left( \frac{(\frac{\mu}{\sigma} L_s)^2}{2} \right) \left[ 1 - \Phi \left( \frac{1 + e}{\sigma} \left( \frac{\mu}{\sigma} L_s \right)^2 \right) \right] - a\alpha \mu L_m \exp \left( \frac{(\frac{\mu}{\sigma} L_m)^2}{2} \right) \left[ 1 + \Phi \left( \frac{1 + e}{\sigma} \left( \frac{\mu}{\sigma} L_m \right)^2 \right) \right] - \left( \frac{e}{1 + e} \right)^{1/2} \frac{2a\alpha \sigma}{\sqrt{2\pi}} \left[ \exp \left( \frac{-((\frac{\mu}{\sigma} L_s)^2)}{2e} \right) - \exp \left( \frac{-((\frac{\mu}{\sigma} L_m)^2)}{2e} \right) \right] > 0.
\]

From (A3), \( g_h(e_{TB}, L_s, L_m) > 0 \). Given (S3), it follows that \( e_h(h) < 0 \) for all \( h > m - \frac{\mu}{a\sigma^2} \). \( Q.E.D. \)
Proof of Corollary 3

Let \( h \in \left[ -\left( s + \frac{\mu}{\alpha \sigma^2} \right), m - \frac{\mu}{\alpha \sigma^2} \right] \). We re-write the function \( J(e, L_s, L_m) \) as:

\[
J(e, L_s, L_m) = \left[ V_e(D,e) - \frac{1}{2(1+e)} \right] \left( \frac{1}{1+e} \right)^{1/2} \left[ \Phi \left( \frac{(\frac{\mu}{\sigma}(L_s))^2}{e} \right) + \Phi \left( \frac{(\frac{\mu}{\sigma}(L_m))^2}{e} \right) \right] \\
+ V_e(D,e) \left\{ \exp \left( \frac{(\frac{\mu}{\sigma}(L_s))^2}{2} \right) \times \left[ 1 - \Phi \left( \frac{(\frac{\mu}{\sigma}(L_s))^2}{e(1+e)} \right) \right] \right\} \\
+ \exp \left( \frac{(\frac{\mu}{\sigma}(L_m))^2}{2} \right) \times \left[ 1 - \Phi \left( \frac{(\frac{\mu}{\sigma}(L_m))^2}{e(1+e)} \right) \right].
\]

Evaluating this function at the second best effort and given (5) we obtain

\[
J(e_{SB}, L_s, L_m) = V_e(D,e_{SB}) \left\{ \exp \left( \frac{(\frac{\mu}{\sigma}(L_s))^2}{2} \right) \times \left[ 1 - \Phi \left( \frac{(\frac{\mu}{\sigma}(L_s))^2}{e_{SB}(1+e_{SB})} \right) \right] \right\} > 0.
\]

This implies that \( E_e U_a(\varphi_a(e_{SB})) = -\exp(-(1/2)\mu^2/\sigma^2 - aF + V(D,e_{SB})) \times J(e_{SB}, L_s, L_m) < 0. \)

Therefore, for the constrained manager, the marginal utility of effort at \( e_{SB} \) is negative. Since \( e_{TB} \) is unique and the function is continuous in \( e \), given conditions (A1) and (A2), it follows that \( e_{SB} > e_{TB} \) for all \( h \in \left[ -\left( s + \frac{\mu}{\alpha \sigma^2} \right), m - \frac{\mu}{\alpha \sigma^2} \right] \). Given Corollary 2, this result holds for all \( h \in \mathbb{R} \). Next we show that

\[
\lim_{z \to \infty} \left[ \exp \left( z/2 \right) \times \left( 1 - \Phi \left( \frac{z(1+e)}{e} \right) \right) \right] = 0. \tag{A7}
\]

Re-writing (A7) and applying L’Hôpital’s rule we get:

\[
\lim_{z \to \infty} \frac{1 - \Phi \left( \frac{z(1+e)}{e} \right)}{\exp \left( -z/2 \right)} = \lim_{z \to \infty} \frac{\exp(-z/2e)}{z^{1/2}} = 0.
\]

Therefore, given (A6) and (A7), \( J(e_{SB}, L_s, L_m) \) tends to zero when \( m \) and \( s \) tend to infinity.

In the limit, the constrained manager’s marginal expected utility of effort becomes zero at \( e_{SB} \), \( E_e U_a(\varphi_a(e_{SB})) = 0 \). Q.E.D.

Proof of Corollary 4

**Lemma 1** For all \( 0 < x < \infty, \frac{1}{2} (1 - \Phi(x)) - \phi(x) < 0. \)

**Proof:** See Lemma 1 in Gómez and Sharma (2006)
Let \( m \to \infty \) and \( 0 \leq s < \infty \). We call \( g_{h}(e, L_{s}) = \lim_{m \to \infty} g_{h}(e, L_{s}, L_{m}) \) and \( g_{eh}(e, L_{s}) = \lim_{m \to \infty} g_{eh}(e, L_{s}, L_{m}) \). From (A5), \( g_{h}(e, L_{s}) < 0 \) for \( h < -(s + \frac{\mu}{\alpha \sigma^{2}}) \). For \( h > -(s + \frac{\mu}{\alpha \sigma^{2}}) \),
\[
g_{h}(e, L_{s}) = 2a_{h}L_{s} \times \exp \left( \frac{(\frac{\mu}{\alpha \sigma^{2}}L_{s})^{2}}{2} \right) \left\{ \frac{1}{2} - \Phi \left( \frac{1 + \gamma \left( \frac{\mu}{\alpha \sigma^{2}}L_{s} \right)^{2} \right) \right\}
\]
\[
- \phi \left( \frac{1 + \gamma \left( \frac{\mu}{\alpha \sigma^{2}}L_{s} \right)^{2} \right) \right\} < 0, \text{ given Lemma 1.}
\]
Therefore, \( g_{h}(e, L_{s}) < 0 \) for all \( h \in \mathbb{R} \). From (A3), \( g_{eh}(e, L_{s}) < 0 \) for all \( h \in \mathbb{R} \). Thus, \( e_{h}(h) > 0 \) for all \( h \in \mathbb{R} \). Following the same procedure, it is trivial to show that \( e_{h}(h) < 0 \) for all \( h \in \mathbb{R} \) when \( s \to \infty \) and \( 1 \leq m < \infty \). Q.E.D.

**Proof of Proposition 3**

Replacing (6) in the investor’s utility function:

\[
EU(\varphi_{h}(y)) = -\exp(b(F - R)) \times \begin{cases} 
\exp \left( -b(h - (1 - \alpha)(s + h))E(x|y) + (b^{2}/2)(h - (1 - \alpha)(s + h))^{2} \text{Var}(x|y) \right) & \text{if } y < -\frac{\mu}{\alpha}L_{s} \\
\exp \left( -b(h + (1 - \alpha)\frac{\mu + \mu y}{\alpha \sigma^{2}})E(x|y) + (b^{2}/2)(h + (1 - \alpha)\frac{\mu + \mu y}{\alpha \sigma^{2}})^{2} \text{Var}(x|y) \right) & \text{otherwise}
\end{cases}
\]

For \( y < -\frac{\mu}{\alpha}L_{s} \), we obtain:

\[
EU(\varphi_{h}(y)) = -\exp(b(F - R) - (1/2)\frac{\mu^{2}}{\sigma^{2}}) \left( \frac{e}{1 + e} \right)^{1/2} \frac{1}{\sqrt{2\pi\sigma}} \times \begin{cases} 
\exp \left( \left( \frac{\mu}{\alpha} + t(\alpha)(L_{s} - 1) - bh \sigma^{2}\mu \right)^{2} \right) & \text{if } y < -\frac{\mu}{\alpha}L_{s} \\
\exp \left( \frac{1}{2(1 + e)} \left( \frac{\mu}{\alpha} t(\alpha) - 1 + bh \sigma^{2}\mu \right)^{2} \right) \exp \left( \frac{\mu^{2}}{2\alpha^{2}} \left( \frac{T(\alpha) - 1 - bh(T(\alpha) - 1)\sigma^{2}\mu}{1 + eT(\alpha)} \right)^{2} \right) & \text{otherwise}
\end{cases}
\]

\[
\exp \left( \left( \frac{\mu}{\alpha} \frac{T(\alpha) - 1 - bh(T(\alpha) - 1)\sigma^{2}\mu}{1 + eT(\alpha)} \right)^{2} \right) \times \exp \left( \frac{\mu}{\alpha} \frac{T(\alpha) - 1 - bh(T(\alpha) - 1)\sigma^{2}\mu}{1 + eT(\alpha)} \right)^{2} \times \end{cases}
\]

If \( y > \frac{\mu}{\alpha}L_{m} \).
Integrating over \( k \)

Replace \( k = \frac{1}{1 + e} \) \( \left\{ \begin{array}{ll} \left( \frac{\theta}{\sigma} - \frac{\mu}{\sigma} \left( 1 + t(\alpha)(L_s - 1) - bh \frac{a^2}{\mu} \right) \right)^2 & \text{if } y < -\frac{\mu}{e} L_s \\
(1 + eT(\alpha)) \left( \frac{\theta}{\sigma} + \frac{T(\alpha) - bh(t(\alpha) - 1)^2}{1 + eT(\alpha)} \right)^2 & \text{otherwise} \end{array} \right. \)

Integrating over \( k \) and given the definition of \( \Phi(\cdot) \), the unconditional utility function becomes

\[
EU_b(\varphi_b(e)) = -\exp(b(F - R) - (1/2)\mu^2/\sigma^2) \times f(e, L_s, L_m) \text{ with } f(e, L_s, L_m) = (1/2) \times
\]

\[
\exp \left( \frac{\left( \frac{\theta}{\sigma} (1 + t(\alpha)(L_s - 1) - bh \frac{a^2}{\mu}) \right)^2}{2} \right) [1 + \Phi \left( \frac{1 + e}{\sigma} \left( \frac{\theta}{\sigma} (1 + \frac{t(\alpha) e}{1 + e} (L_s - 1) - \frac{\mu}{1 + e} bh \frac{a^2}{\mu}) \right)^2 \right) ] +
\]

\[
\exp \left( \frac{1}{2} \frac{1}{1 + e} \left( \frac{\theta}{\sigma} \left( t(\alpha) - 1 bh \frac{a^2}{\mu} \right) \right)^2 \right) \exp \left( \frac{\sigma^2}{2 + \frac{1}{e}} \frac{T(\alpha) - bh(t(\alpha) - 1) \frac{a^2}{\mu}}{1 + eT(\alpha)} \right) \left( 1 + eT(\alpha) \right)^{1/2}
\]

\[
\Phi \left( \frac{1 + e}{\sigma} \left( \frac{\theta}{\sigma} (1 + \frac{t(\alpha) e}{1 + e} (L_s - 1) - \frac{\mu}{1 + e} bh \frac{a^2}{\mu}) \right)^2 \right) -
\]

\[
\exp \left( \frac{\left( \frac{\theta}{\sigma} (t(\alpha)(1 + L_m) - 1 bh \frac{a^2}{\mu}) \right)^2}{2} \right) \left[ 1 + \Phi \left( \frac{1 + e}{\sigma} \left( \frac{\theta}{\sigma} (1 + \frac{t(\alpha) e}{1 + e} (1 + L_m) - 1 + \frac{\mu}{1 + e} bh \frac{a^2}{\mu}) \right)^2 \right) \right].
\]

if \( h < -\left( \frac{a^2 + bh(1 - \alpha)}{\alpha(a + \beta)} + (1 + e)\frac{\mu}{\alpha(a + \beta)\sigma^2} \right) \)

\[
\exp \left( \frac{\left( \frac{\theta}{\sigma} (1 + t(\alpha)(L_s - 1) - bh \frac{a^2}{\mu}) \right)^2}{2} \right) [1 + \Phi \left( \frac{1 + e}{\sigma} \left( \frac{\theta}{\sigma} (1 + \frac{t(\alpha) e}{1 + e} (L_s - 1) - \frac{\mu}{1 + e} bh \frac{a^2}{\mu}) \right)^2 \right) ] +
\]

\[
\exp \left( \frac{1}{2} \frac{1}{1 + e} \left( \frac{\theta}{\sigma} \left( t(\alpha) - 1 bh \frac{a^2}{\mu} \right) \right)^2 \right) \exp \left( \frac{\sigma^2}{2 + \frac{1}{e}} \frac{T(\alpha) - bh(t(\alpha) - 1) \frac{a^2}{\mu}}{1 + eT(\alpha)} \right) \left( 1 + eT(\alpha) \right)^{1/2}
\]

\[
\Phi \left( \frac{1 + e}{\sigma} \left( \frac{\theta}{\sigma} (1 + \frac{t(\alpha) e}{1 + e} (1 + L_m) - 1 + \frac{\mu}{1 + e} bh \frac{a^2}{\mu}) \right)^2 \right) -
\]

\[
\exp \left( \frac{\left( \frac{\theta}{\sigma} (t(\alpha)(1 + L_m) - 1 bh \frac{a^2}{\mu}) \right)^2}{2} \right) \left[ 1 + \Phi \left( \frac{1 + e}{\sigma} \left( \frac{\theta}{\sigma} (1 + \frac{t(\alpha) e}{1 + e} (1 + L_m) - 1 + \frac{\mu}{1 + e} bh \frac{a^2}{\mu}) \right)^2 \right) \right].
\]
\[
\exp\left(\frac{1}{2} \frac{1}{1+e^t} \left(\frac{a}{e} \left(1 + \frac{1}{1+e^t} \right)^2 \right)\right) [1 - \Phi \left(\frac{1 + \frac{1}{1+e^t} \left(1 - \frac{e}{\tau + e} bh \frac{\alpha^2}{\mu}\right)}{\sqrt{2}}\right)] + \\
\exp\left(\frac{1}{2} \frac{1}{1+e^t} \left(\frac{a}{e} \left(1 + bh \frac{\alpha^2}{\mu}\right)^2\right)\right) \exp\left(\frac{\mu^2}{2} \frac{e}{1+e^t} \left(\frac{\tau(t) - \frac{1}{1+e^t} bh \delta + \frac{\beta^2}{\mu}\right)^2\right) \right) \left(\frac{1}{1+e^t}\right)^{1/2} \\
\left[\frac{1 + \frac{1}{1+e^t} \left(1 - bh \frac{\alpha^2}{\mu}\right)}{\sqrt{2}}\right] + \\
\Phi \left(\frac{1 + \frac{1}{1+e^t} \left(1 - bh \frac{\alpha^2}{\mu}\right)}{\sqrt{2}}\right) \left(1 + L_m\right) - 1 - \frac{e}{\tau + e} bh \delta \frac{\alpha^2}{\mu}\right)^2\right]\] + \\
\exp\left(\frac{1}{2} \frac{1}{1+e^t} \left(\frac{a}{e} \left(1 + bh \frac{\alpha^2}{\mu}\right)^2\right)\right) \exp\left(\frac{\mu^2}{2} \frac{e}{1+e^t} \left(\frac{\tau(t) - \frac{1}{1+e^t} bh \delta + \frac{\beta^2}{\mu}\right)^2\right) \right) \left(\frac{1}{1+e^t}\right)^{1/2} \\
\left[\frac{1 + \frac{1}{1+e^t} \left(1 - bh \frac{\alpha^2}{\mu}\right)}{\sqrt{2}}\right] + \\
\Phi \left(\frac{1 + \frac{1}{1+e^t} \left(1 - bh \frac{\alpha^2}{\mu}\right)}{\sqrt{2}}\right) \left(1 + L_m\right) - 1 - \frac{e}{\tau + e} bh \delta \frac{\alpha^2}{\mu}\right)^2\right]\]

if \(-\frac{a + eb(1 - \alpha(2 - \tau(t)))}{\alpha + eb(1 - \alpha(2 - \tau(t)))}\leq h < \frac{a + eb(1 - \alpha(2 - \tau(t)))}{\alpha + eb(1 - \alpha(2 - \tau(t)))}\frac{(1+\mu)}{(1+\mu)}\) \leq h \leq \frac{a + eb(1 - \alpha(2 - \tau(t)))}{\alpha + eb(1 - \alpha(2 - \tau(t)))}\frac{(1+\mu)}{(1+\mu)}\)

\[
\exp\left(\frac{1}{2} \frac{1}{1+e^t} \left(\frac{a}{e} \left(1 + bh \frac{\alpha^2}{\mu}\right)^2\right)\right) \exp\left(\frac{\mu^2}{2} \frac{e}{1+e^t} \left(\frac{\tau(t) - \frac{1}{1+e^t} bh \delta + \frac{\beta^2}{\mu}\right)^2\right) \right) \left(\frac{1}{1+e^t}\right)^{1/2} \\
\left[\frac{1 + \frac{1}{1+e^t} \left(1 - bh \frac{\alpha^2}{\mu}\right)}{\sqrt{2}}\right] + \\
\Phi \left(\frac{1 + \frac{1}{1+e^t} \left(1 - bh \frac{\alpha^2}{\mu}\right)}{\sqrt{2}}\right) \left(1 + L_m\right) - 1 - \frac{e}{\tau + e} bh \delta \frac{\alpha^2}{\mu}\right)^2\right]\]

if \(\frac{a + eb(1 - \alpha(2 - \tau(t)))}{\alpha + eb(1 - \alpha(2 - \tau(t)))}\frac{(1+\mu)}{(1+\mu)}\) \leq h \leq \frac{a + eb(1 - \alpha(2 - \tau(t)))}{\alpha + eb(1 - \alpha(2 - \tau(t)))}\frac{(1+\mu)}{(1+\mu)}\)

\[
\exp\left(\frac{1}{2} \frac{1}{1+e^t} \left(\frac{a}{e} \left(1 + bh \frac{\alpha^2}{\mu}\right)^2\right)\right) \exp\left(\frac{\mu^2}{2} \frac{e}{1+e^t} \left(\frac{\tau(t) - \frac{1}{1+e^t} bh \delta + \frac{\beta^2}{\mu}\right)^2\right) \right) \left(\frac{1}{1+e^t}\right)^{1/2} \\
\left[\frac{1 + \frac{1}{1+e^t} \left(1 - bh \frac{\alpha^2}{\mu}\right)}{\sqrt{2}}\right] + \\
\Phi \left(\frac{1 + \frac{1}{1+e^t} \left(1 - bh \frac{\alpha^2}{\mu}\right)}{\sqrt{2}}\right) \left(1 + L_m\right) - 1 - \frac{e}{\tau + e} bh \delta \frac{\alpha^2}{\mu}\right)^2\right]\]
\[
\text{if } h > m \frac{a + e(h(1-\alpha))}{\alpha(a-e\beta)} - \frac{(1+e)\mu}{\alpha(a-e\beta)^2}. \quad Q.E.D.
\]

Proof of Proposition 4

Assume first that the manager’s effort choice is publicly observable. Given equation (11), the investor chooses the contract \((\alpha, h)\) that satisfies the first order optimality condition:

\[
\frac{\partial}{\partial h} EU_b(\varphi_b(e) | \alpha, h) = -\exp(-bR - (1/2)(\mu/\sigma)^2 + (b/\alpha)V(D, e)) \times
\left( \frac{b}{a} g(e, L_s, L_m)^{b/a-1} g_l(e, L_s, L_m) f(e, L_s, L_m) + g(e, L_s, L_m)^{b/a} f_l(e, L_s, L_m) \right) = 0,
\]

for \(i = \{\alpha, h\}\). We distinguish two cases: with and without portfolio constraints.

Without portfolio constraints, \(s \to \infty\) and \(m \to \infty\). The manager’s expected utility (4) is independent of \(\alpha\) and \(h\). The investor’s expected utility in (A8) becomes:

\[
f(e) = \exp \left( \frac{1}{2} \frac{1}{1 + e} \left( \frac{\mu}{\sigma} \left( t(\alpha) - 1 + bh \frac{\sigma^2}{\mu} \right) \right)^2 \right)
\]

\[
\exp \left( \frac{\mu^2}{2\sigma^2} \frac{e}{1 + e} \left( \frac{T(\alpha) - 1 - bh(t(\alpha) - 1) \frac{\sigma^2}{\mu}}{1 + eT(\alpha)} \right)^2 \right) \left( \frac{1}{1 + T(\alpha)e} \right)^{1/2}.
\]

By definition, \(t(\alpha_{FB}) = T(\alpha_{FB}) = 1; t_\alpha(\alpha_{FB}) = \frac{a+b}{ab}; T_\alpha(\alpha_{FB}) = 0.\) Then, it follows immediately that \(f_i(e, \alpha_{FB}, 0) = 0, i = \{\alpha, h\},\) for any effort \(e.\) Hence, the contract \((\alpha_{FB}, 0)\) is (first order condition) optimal.

With portfolio constraints, notice first that \(g(e, L_s, L_m | \alpha_{FB}, 0) = f(e, L_s, L_m | \alpha_{FB}, 0).\) Let us analyze now the partial derivatives of function \(f\) and \(g\) with respect to \(\alpha\) and \(h.\) Taking the derivative of (A8) with respect to \(h\) and evaluating it at the contract \((\alpha_{FB}, 0)\) yields:

\[
f_h(e, L_s, L_m | \alpha_{FB}, 0) = -2 \frac{b^2}{a + b}\mu
\]

\[
\left\{ \exp \left( \frac{\mu L_s(0)^2}{2} \right) L_s(0) \times
\left( \frac{1}{2} \left[ 1 - \Phi \left( \frac{1 + e}{\nu} \left( \frac{\mu}{\sigma} L_s(0)^2 \right) \right) \right] - \phi \left( \frac{1 + e}{\nu} \left( \frac{\mu}{\sigma} L_s(0)^2 \right) \right) \right)
\]

\[
\exp \left( \frac{\mu L_m(0)^2}{2} \right) L_m(0) \times
\left( \frac{1}{2} \left[ 1 - \Phi \left( \frac{1 + e}{\nu} \left( \frac{\mu}{\sigma} L_m(0)^2 \right) \right) \right] - \phi \left( \frac{1 + e}{\nu} \left( \frac{\mu}{\sigma} L_m(0)^2 \right) \right) \right) \right\}.
\]

Equation (A4) evaluated at \((\alpha_{FB}, 0)\) becomes:
\[ g_h(e, L_s, L_m | \alpha_{FB}, 0) = 2 \frac{ab}{a + b} \mu \]
\[ \left\{ \exp \left( \frac{\left( \frac{\mu}{\sigma} L_s(0) \right)^2}{2} \right) L_s(0) \times ight. \\
\left. \left( \frac{1}{2} \left[ 1 - \Phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_s(0) \right)^2 \right) \right] - \phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_s(0) \right)^2 \right) \right) + \\
\exp \left( \frac{\left( \frac{\mu}{\sigma} L_m(0) \right)^2}{2} \right) L_m(0) \times \\
\left( \frac{1}{2} \left[ 1 - \Phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_m(0) \right)^2 \right) \right] - \phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_m(0) \right)^2 \right) \right) \right\}. \]

Taking the derivative of (A8) with respect to \( \alpha \) and evaluating it at the contract \((\alpha_{FB}, 0)\), we obtain:

\[ f_\alpha(e, L_s, L_m | \alpha_{FB}, 0) = \\
-2bs\mu \exp \left( \frac{\left( \frac{\mu}{\sigma} L_s(0) \right)^2}{2} \right) L_s(0) \times \\
\left( \frac{1}{2} \left[ 1 - \Phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_s(0) \right)^2 \right) \right] - \phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_s(0) \right)^2 \right) \right) \]
\[ -2bm\mu \exp \left( \frac{\left( \frac{\mu}{\sigma} L_m(0) \right)^2}{2} \right) L_m(0) \times \\
\left( \frac{1}{2} \left[ 1 - \Phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_m(0) \right)^2 \right) \right] - \phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_m(0) \right)^2 \right) \right). \]

Equation (A4) at \((\alpha_{FB}, 0)\) can be rewritten as follows:

\[ g_\alpha(e, L_s, L_m | \alpha_{FB}, 0) = \\
2as\mu \exp \left( \frac{\left( \frac{\mu}{\sigma} L_s(0) \right)^2}{2} \right) L_s(0) \times \\
\left( \frac{1}{2} \left[ 1 - \Phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_s(0) \right)^2 \right) \right] - \phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_s(0) \right)^2 \right) \right) + \\
2am\mu \exp \left( \frac{\left( \frac{\mu}{\sigma} L_m(0) \right)^2}{2} \right) L_m(0) \times \\
\left( \frac{1}{2} \left[ 1 - \Phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_m(0) \right)^2 \right) \right] - \phi \left( \frac{1 + e}{e} \left( \frac{\mu}{\sigma} L_m(0) \right)^2 \right) \right). \quad (A10) \]

From the former equations, \( g_i(e, L_s, L_m | \alpha_{FB}, 0) = -\frac{a}{b} f_i(e, L_s, L_m | \alpha_{FB}, 0) \) for \( i = \{\alpha, h\} \).
Evaluating the optimality condition (A9) at \((\alpha_{FB}, 0)\) and given the partial derivatives for \( f \) and \( g \), it follows that the contract \((\alpha_{FB}, 0)\) satisfies the first order optimality condition in the absence of moral hazard.
We turn now to the case of moral hazard. Without portfolio constraints (second best scenario), the manager’s effort (5) is independent of $\alpha$ and $h$. Hence, as we just showed, the contract ($\alpha_{FB}, 0$) is optimal. Under portfolio constraints (third best scenario), the third best effort, $e_{TB}$, is a function of $\alpha$ and $h$. The first order condition for optimality requires that

$$\frac{\partial}{\partial \alpha} EU_b(\varphi_b(e_{TB})|\alpha, h) = \frac{\partial}{\partial e} EU_b(\varphi_b(e_{TB})|\alpha, h) + \frac{\partial}{\partial e} EU_b(\varphi_b(e)|\alpha, h) \frac{\partial}{\partial e} e_{TB}(\alpha, h) = 0,$$

for $i = \{\alpha, h\}$. We have just proved that $\frac{\partial}{\partial \alpha} EU_b(\varphi_b(\alpha)|\alpha_{FB}, 0) = 0$ for all effort. By definition, $\frac{\partial}{\partial \alpha} EU_b(\varphi_b(\alpha)|\alpha_{FB}, 0)_{e=e_{TB}} = -\exp(-bR - (1/2)(\mu/\sigma)^2 + (b/a)V(D, e_{TB})) \frac{\partial}{\partial \alpha} g(e_{TB}, L_s(0), L_m(0))^{b/a} \times [\psi(e_{TB}, L_s(0), L_m(0)) + g(e_{TB}, L_s(0), L_m(0))].$ Given (A1), the later equation can be rewritten as: $-\exp(-bR - (1/2)(\mu/\sigma)^2 + (b/a)V(D, e_{TB})) g(e_{TB}, L_s(0), L_m(0))^{b/a} g(e_{TB}, L_s(0), L_m(0)) > 0$ given Proposition 1 and Corollary 1.

From Proposition 2 and Corollary 2, for all $\alpha \in (0, 1], \frac{\partial}{\partial \alpha} e_{TB}(\alpha, h) > 0 ( < 0)$ for $h < h^*$ ($h > h^*$); $\frac{\partial}{\partial \alpha} e_{TB}(\alpha, h) = 0$ for $h = h^*$. Hence, $\frac{\partial}{\partial \alpha} e_{TB}(\alpha_{FB}, 0) = 0$ only for $h^* = 0$. We investigate now whether $\frac{\partial}{\partial \alpha} e_{TB}(\alpha_{FB}, h^* = 0) = 0$.

$$J_{\alpha}(e_{TB}, L_s, L_m|\alpha_{FB}, h^* = 0) = V_e(D, e_{TB}) g_\alpha(e_{TB}, L_s, L_m|\alpha_{FB}, h^* = 0) + g_\alpha(e_{TB}, L_s, L_m|\alpha_{FB}, h^* = 0) < 0.$$ From Assumption (S3), $V_e(D, e_{TB}) > 0$. From (A11) and given Lemma 1, $g_\alpha(e_{TB}, L_s, L_m|\alpha_{FB}, h^* = 0) < 0$. Given (A2), $J_{\alpha}^{-1}(e_{TB}, L_s, L_m|\alpha_{FB}, h^* = 0) > 0$. Therefore, $\frac{\partial}{\partial \alpha} e_{TB}(\alpha_{FB}, h^* = 0) = -J_{\alpha}(e_{TB}, L_s, L_m|\alpha_{FB}, h^* = 0) \times J_{\alpha}^{-1}(e_{TB}, L_s, L_m|\alpha_{FB}, h^* = 0) > 0$. Thus, the contract ($\alpha_{FB}, 0$) is suboptimal. Q.E.D.

When we introduce selectivity information as in section 7, the first order condition (A9) under public information becomes:

$$\frac{\partial}{\partial \alpha} EU_b(\varphi_b(\alpha)|\alpha, h) = -\exp(-bR - (1/2)(\mu/\sigma)^2 + (b/a)V(D, e)) \times \left[ (\frac{1}{2} g(e_z, L_s, L_m)^{b/a} g_\alpha(e_z, L_s, L_m) f(e_z, L_s, L_m) + g_\alpha(e_z, L_s, L_m)^{b/a} f_\beta(e_z, L_s, L_m) \right] g(e_z, s, m)^{b/a} f(e_z, s, m) + \left( \frac{1}{2} g(e_z, s, m)^{b/a} g_\alpha(e_z, s, m) f(e_z, s, m) + g_\alpha(e_z, s, m)^{b/a} f_\beta(e_z, s, m) \right) g(e_z, L_s, L_m)^{b/a} f(e_z, L_s, L_m) = 0,$$

for $i = \{\alpha, h\}$. Notice that $g_\alpha(e_z, s, m) = f_\beta(e_z, s, m) = 0$. Moreover, $g(e_z, s, m|\alpha_{FB}) = f(e_z, s, m|\alpha_{FB})$. Hence, the same arguments used above to prove the (first order) optimality of the contract ($\alpha_{FB}, 0$) when effort is observable hold, both with and without portfolio constraints, in the presence of selectivity information.

In the case of moral hazard, when both effort choices are not observable, the first order condition (A11) becomes:

$$\frac{\partial}{\partial \alpha} EU_b(\varphi_b(e_{TB})|\alpha, h) = \frac{\partial}{\partial e} EU_b(\varphi_b(e_{TB})|\alpha, h) + \frac{\partial}{\partial e} EU_b(\varphi_b(e)|\alpha, h) \left( \frac{\partial}{\partial e} e_{TB}(\alpha, h), \frac{\partial}{\partial e} e_{TB}(\alpha, h) \right) = 0,$$

for $i = \{\alpha, h\}$ and $e_{TB} = (e_{TB}, e_{TB})$. Evaluated at the first best contract ($\alpha_{FB}, 0$),
\[
\frac{\partial}{\partial \theta} \text{EU}_b(\varphi_b(e_{TB}))(\alpha, h) = \\
- \exp\left(-bR - \frac{1}{2}(\mu/\sigma_x)^2 + (b/a)V(D, e_{TB})g(e_{TB}, L_s, L_m)^{b/a}g(e_{TB}, s, m)^{a/b}\right) \\
\left[g_x(e_{TB}, L_s, L_m)g(e_{TB}, s, m)\frac{\partial}{\partial \nu}e_{TB}(\alpha, h) + g_x(e_{TB}, s, m)g(e_{TB}, L_s, L_m)\frac{\partial}{\partial \nu}e_{TB}(\alpha)\right].
\]

Following the same arguments as in the case without selectivity we conclude that the first best contract is suboptimal. Q.E.D.
Table 1: Optimal contract \((\alpha_{TB}, h_{TB})\) for different values of the maximum long \((m)\) and short \((s)\) position on the market portfolio allowed to the manager. \(\alpha\) is the percentage incentive fee; \(h\) is the percentage of the benchmark portfolio invested in the market portfolio. \(m = 1\) and \(s = 0\) imply zero leverage. \(\alpha\) represents the manager’s risk aversion coefficient. The investor’s risk aversion coefficient is \(b = 4\). The first best incentive fee is \(\alpha_{FB} = 50\%\) for \(a = 4\); \(\alpha_{FB} = 33\%\) for \(a = 8\); \(\alpha_{FB} = 16\%\) for \(a = 20\).
### Panel A
#### Third Best Effort ($e_{TB}$)

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### Panel B
#### $e_{TB}/e(\alpha_{FB}, 0) - 1$

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Table 2: Panel A shows the percentage third best effort $e_{TB}$ for different values of the maximum long ($m$) and short ($s$) position on the market portfolio allowed to the manager. $m = 1$ and $s = 0$ imply zero leverage. $a$ represents the manager’s risk aversion coefficient. The investor’s risk aversion coefficient is $b = 4$. The first best effort is 61.8% ($a = 4$), 82.29% ($a = 8$), 130.28% ($a = 20$), respectively. The second best effort is 36.63%, independently of $a$. Panel B shows, for every contract and portfolio constraint, the percentage change in the third best optimal effort in Panel A with respect to the effort the manager would exert if compensated with the suboptimal first best contract, $(\alpha_{FB}, 0)$. 
Table 3: This table presents the CEW loss that would result for the investor from offering the manager a suboptimal first best contract ($\alpha_{FB}, 0$). The figures could be interpreted as the annual excess risk-free return (above the bond’s return, $R$) that would compensate the investor for the loss in expected utility from the suboptimal first best contract. The investor’s risk aversion is $b = 4$. 

<table>
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We assume that short-selling is totally forbidden \((s = 0)\) and there is no limit to margin purchase \((m \to \infty)\). For simplicity, let \(\alpha = 1\). After putting effort \(e\) the manager receives a signal \(y\) and makes her optimal portfolio \(\theta\). When \(h = 0\) (bottom portfolio line), all signals \(y < -\frac{\mu}{e}\) lead to short-selling. When \(h > 0\) (upper portfolio line), the short-selling bound is hit for signals \(y < -\frac{\mu}{e} L_s\). In both cases, the region of these non-implementable portfolios is marked by the thick line. Under benchmarking \((h > 0)\) there is an incremental area for implementable signals relative to the case of no benchmarking. The size of this area, \(\frac{h\alpha}{e/\sigma^2}\), increases with benchmarking \((h)\) and the manager’s risk aversion \((\alpha)\); it has probability mass equal to the shaded area in the density function plot.

Figure 1: We assume that short-selling is totally forbidden \((s = 0)\) and there is no limit to margin purchase \((m \to \infty)\). For simplicity, let \(\alpha = 1\). After putting effort \(e\) the manager receives a signal \(y\) and makes her optimal portfolio \(\theta\). When \(h = 0\) (bottom portfolio line), all signals \(y < -\frac{\mu}{e}\) lead to short-selling. When \(h > 0\) (upper portfolio line), the short-selling bound is hit for signals \(y < -\frac{\mu}{e} L_s\). In both cases, the region of these non-implementable portfolios is marked by the thick line. Under benchmarking \((h > 0)\) there is an incremental area for implementable signals relative to the case of no benchmarking. The size of this area, \(\frac{h\alpha}{e/\sigma^2}\), increases with benchmarking \((h)\) and the manager’s risk aversion \((\alpha)\); it has probability mass equal to the shaded area in the density function plot.
Figure 2: Portfolio choice is constrained. The “absolute” capital market line represents the portfolio choice problem of an uninformed investor ($e = 0$) that maximizes his expected utility. Notice that the slope of the capital market line coincides with the market Sharpe ratio, $\mu/\sigma$. His preferences are represented by the indifference curve $U(\theta)$. In the example, his optimal portfolio portfolio holds less than 50% in the market. If the investor were given a benchmark $h > 0$ then he will choose a tangent portfolio $\tilde{\theta} = \theta - h^*$ in the “relative” capital market line that trades off excess expected return $\tilde{\theta} \mu$ against tracking error standard deviation, $\tilde{\theta} \sigma$. Notice that given the portfolio constraints, for $h = h^*$ the investor’s optimal unconditional portfolio is equidistant from either boundary. The Information Ratio (excess return over the benchmark relative to the portfolio tracking error’s volatility) does not change and it is equal to the Sharpe Ratio. If the portfolio decision is taken by a manager with the ability to increase the precision of her private signal by putting some effort then, moving $h$ from zero to $h^*$ results in the highest effort expenditure for the constrained manager. The Information Ratio (the slope) increases to $IR(e) > (\mu/\sigma)\sqrt{1 + e}$. The manager’s optimal portfolio would be greater than 50% (the tangency portfolio for $U(\tilde{\theta})$).
Figure 3: This figure represents the Information Ratio as a function of the signal $y$ and given effort $e$. Notice that, when $e = 0$, the Information Ratio coincides with the Sharpe Ratio for all signal $y$. When $e$ increases the slope increases in absolute value, making the Information Ratio higher for all signal $y$. As $e \to \infty$, in the limit, the Information Ratio also tends to infinity. For $y = \mu$, the Information Ratio becomes $\frac{\mu}{\sigma} \sqrt{1 + e}$. Averaging across $y$, the expected Information Ratio is higher than $\frac{\mu}{\sigma} \sqrt{1 + e}$ since for all $y < -\frac{\mu}{e}$, the Information Ratio “bounces back” the manager would short the risky asset.
Figure 4: The manager is totally constrained in her portfolio choice: $m = 1$ and $s = 0$. The vertical axis in each figure represents the investor’s expected utility when the manager’s effort choice is observable (Panel A) and under moral hazard (Panel B). The maximum and minimum expected utility within the values of the contract represented are reported. The horizontal axes represent the incentive fee, $\alpha$, and the percentage in the benchmark portfolio invested in the market, $h$, respectively. The three-dimensional cross identifies the optimal contract. $b$ ($a$) denotes the investor’s (manager’s) risk aversion coefficient.
Figure 5: Each figure represents the percentage variation in effort, relative to the third best, when we change the benchmark composition $h$ holding constant the optimal third best incentive fee, $\alpha_{TB}$, for values of $a = 4$, $a = 8$ and $a = 20$, respectively. The investor’s risk aversion is $b = 4$. The manager is fully constrained: $m = 1$ and $s = 0$. $h^*$ is the effort maximizing benchmark composition in the manager’s partial equilibrium problem. $h_{TB}$ is the third best optimal benchmark composition. For $a = 20$, the third best optimal benchmark, $h_{TB} = 20\%$, and $h^* = 16.33\%$. 
Figure 6: Each figure represents the percentage loss in certainty equivalent wealth (CEW), relative to the third best, when we change the benchmark composition $h$ holding constant the optimal third best incentive fee, $\alpha_{TB}$, for values of $a = 4$, $a = 8$ and $a = 20$, respectively. The investor’s risk aversion is $b = 4$. The manager is fully constrained: $m = 1$ and $s = 0$. $h_{TB}$ is the third best optimal benchmark composition.