The quasitopological fundamental group and the first shape map

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Introduction

Joint with Paul Fabel.

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The fundamental group $\pi_1(X, x_0)$ of a Peano continuum $X, x_0 \in X$ is either

- finitely presented (when $X$ has a universal covering)
- or uncountable (when $X$ does not have a universal covering)
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Motivation/Application:
- Distinguish homotopy types
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- Distinguish homotopy types
- Provides new direction for combinatorial theory of infinitely generated groups, i.e. slender/n-slender/n-cotorsion free groups (Eda, Fischer)
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Motivation/Application:

- Distinguish homotopy types
- Provides new direction for combinatorial theory of infinitely generated groups, i.e. slender/n-slender/n-cotorsion free groups (Eda, Fischer)
- Natural topologies on homotopical invariants provide (wild) geometric models for objects in topological algebra.
The Hawaiian earring $\mathbb{H}$
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The homomorphisms $\pi_1(\mathbb{H}, 0) \to \pi_1\left(\bigvee_{i=1}^n S^1, 0\right) = F(x_1, \ldots, x_n)$ induce a canonical homomorphism

$$\Psi : \pi_1(\mathbb{H}, 0) \to \lim_{\leftarrow \atop n} F(x_1, \ldots, x_n)$$
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**Theorem (Griffiths, Morgan, Morrison):** $\ker \Psi = 1$ so $\Psi$ is injective.
The homomorphisms \( \pi_1(\mathbb{H}, 0) \to \pi_1 \left( \bigvee_{i=1}^n S^1, 0 \right) = F(x_1, ..., x_n) \) induce a canonical homomorphism

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\]

**Theorem (Griffiths, Morgan, Morrison):** \( \ker \Psi = 1 \) so \( \Psi \) is injective. An element in \( \pi_1(\mathbb{H}, 0) = \text{Im}(\Psi) \) is a sequence \( (w_1, w_2, ...) \) where \( w_n \in F(x_1, ..., x_n) \) and for every fixed generator \( x_i \) the number of times \( x_i \) appears in \( w_n \) is eventually constant.
The Čech expansion

Choose a finite open cover $\mathcal{U}_n$ of $X$ consisting of path connected open balls $U$ with $diam(U) < \frac{1}{n}$ such that $\mathcal{U}_{n+1} \succeq \mathcal{U}_n$ (refinement).
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Choose a finite open cover $\mathscr{U}_n$ of $X$ consisting of path connected open balls $U$ with $\text{diam}(U) < \frac{1}{n}$ such that $\mathscr{U}_{n+1} \supseteq \mathscr{U}_n$ (refinement). Let $X_n = N(\mathscr{U}_n)$ be the nerve of $\mathscr{U}_n$.

Refinement gives an inverse sequence of polyhedra

$$
\cdots \longrightarrow X_{n+1} \xrightarrow{p_{n+1,n}} X_n \xrightarrow{p_{n,n-1}} \cdots \longrightarrow X_2 \xrightarrow{p_{2,1}} X_1
$$
The fundamental pro-group

The fundamental pro-group is the inverse sequence \((\pi_1(X_n, x_n), (p_{n+1,n})_*)\) of finitely generated groups.
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The first shape homotopy group is $\tilde{\pi}_1(X, x_0) = \limleftarrow(\pi_1(X_n, x_n), (p_{n+1,n})_*)$. 
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Using partitions of unity, construct canonical maps \(p_n : X \rightarrow X_n\) such that \(p_{n+1,n} \circ p_{n+1} \simeq p_n\).
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\[
\begin{array}{cccc}
\pi_1(X_n, x_n) & \xrightarrow{(p_n)_*} & \pi_1(X, x_0) & \xrightarrow{(p_1)_*} \\
\xleftarrow{(p_{n+1,n})_*} & & \xleftarrow{(p_2)_*} & \\
\cdots & \xrightarrow{(p_{n+1,n})_*} & \cdots & \xrightarrow{(p_2,1)_*} \\
& & & \pi_1(X_1, x_1) \\
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The first shape homomorphism is the canonical homomorphism \(\psi : \pi_1(X, x_0) \to \tilde{\pi}_1(X, x_0).\)
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If \(\ker \psi = 1\), we say \(X\) is \(\pi_1\)-shape injective.
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If \(\text{ker } \Psi = 1\), we say \(X\) is \(\pi_1\)-shape injective. e.g. 1-dimensional, planar Peano continua.
The quasitopological fundamental group $\pi_{1}^{qtop}(X, x_{0})$ is the usual fundamental group endowed with the quotient topology w.r.t. $\Omega(X, x_{0}) \to \pi_{1}(X, x_{0})$, $\alpha \to [\alpha]$.

- Discrete iff $X$ admits a universal covering (Fabel).
- $\pi_{1}^{qtop}(X, x_{0})$ can fail to be a topological group, e.g. $\mathbb{I}H$ (Fabel).
- $\pi_{1}^{qtop}(X, x_{0})$ is a quasitopological group.
- A necessary intermediate for a group topology on $\pi_{1}(X, x_{0})$ which has application to the general theory of topological groups, e.g. Every open subgroup of a free topological group is free topological (B).
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Topologizing $\pi_1$
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**Guiding principle:** If $\alpha_n \to \alpha$ in $\Omega(X, x_0)$, then $[\alpha_n] \to [\alpha]$ in $\pi_1^{qtop}(X, x_0)$. 
Open subgroups and invariant separation

We consider separation axioms and other separation properties.
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**Definition:** A space $A$ is **totally separated** if whenever $a \neq b$, there is a clopen set $U \subset A$ with $a \in U$ and $b \notin U$. 

**Remark:** $G$ is invariantly separated $\iff \bigcap N = 1.$ invariantly separated $\Rightarrow$ totally separated $\Rightarrow$ Hausdorff.
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**Definition:** A quasitopological group $G$ is **invariantly separated** if whenever $g \neq h$, there is an open normal subgroup $N \subset G$ such that $gN \neq hN$. 

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**Invariantly separated $\Rightarrow$ totally separated $\Rightarrow$ Hausdorff**
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**Remark:** $G$ is invariantly separated $\iff \bigcap_{N \leq G \text{ open}} N = 1$.

invariantly separated $\Rightarrow$ totally separated $\Rightarrow$ Hausdorff
Comparing the approaches

1. Shape theory $\psi : \pi_1(X, x_0) \to \tilde{\pi}_1(X, x_0)$,

2. Topological separation in $\pi^q_{top}(X, x_0)$.

**Question:** How much of $\pi_1(X, x_0)$ does each method retain (or forget)?
Comparing the approaches

1. Shape theory, \( \psi : \pi_1(X, x_0) \to \tilde{\pi}_1(X, x_0) \),
2. Classical covering maps \( p : Y \to X \),
3. Topological separation in \( \pi_{1}^{qtop}(X, x_0) \).

**Question:** How much of \( \pi_1(X, x_0) \) does each method retain (or forget)?
Spanier groups

Definition:

The Spanier group of \( X \) with respect to \( \mathcal{U}_n \) is the normal subgroup

\[
\pi^{sp}(\mathcal{U}_n, x_0) = \langle [\alpha \cdot \gamma \cdot \alpha^{-1}] | \text{Im}(\gamma) \subset U, U \in \mathcal{U}_n \rangle.
\]

Remark: \( \pi^{sp}(\mathcal{U}_{n+1}, x_0) \subset \pi^{sp}(\mathcal{U}_n, x_0) \), \( n \geq 1 \)

The Spanier group of \( X \) is

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The Spanier group of $X$ with respect to $\mathcal{U}_n$ is the normal subgroup

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Remark: $\pi^{sp}(\mathcal{U}_{n+1}, x_0) \subset \pi^{sp}(\mathcal{U}_n, x_0), n \geq 1$

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Spanier groups

**Definition:**

The **Spanier group of** \( X \) **with respect to** \( \mathcal{U}_n \) **is** the **normal** subgroup

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\]

**Remark:** \( \pi^{sp}(\mathcal{U}_{n+1}, x_0) \subset \pi^{sp}(\mathcal{U}_n, x_0), \ n \geq 1 \)

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Spanier groups

Utility: Spanier groups provide a way to determine when (classical) covering maps exist.

Theorem (Spanier): Given \( H \leq \pi_1(X, x_0) \),

there is a covering map
\[
p : Y \to X, \; p(y_0) = x_0 \quad \iff \quad \pi^{sp}(\mathcal{U}_n, x_0) \subseteq H \text{ for some } n \geq 1
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such that \( p_*(\pi_1(Y, y_0)) = H \)
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such that $p_*(\pi_1(Y, y_0)) = H$

Corollary: $\pi^{sp}(X, x_0)$ consists precisely of the homotopy classes $[\alpha] \in \pi_1(X, x_0)$ for which $\alpha$ lifts to a loop for every covering $p : (Y, y_0) \rightarrow (X, x_0)$, i.e.

\[
\pi^{sp}(X, x_0) = \bigcap_{n \geq 1} \pi^{sp}(\mathcal{U}_n, x_0) = \bigcap_{p : (Y, y_0) \rightarrow (X, x_0) \text{ covering}} p_*(\pi_1(Y, y_0))
\]
**Definition:** The **thick Spanier group** of $X$ with respect to $\mathcal{U}_n$ is the *normal* subgroup
\[
\Pi^{sp}(\mathcal{U}_n, x_0) = \langle [\alpha \cdot \gamma_1 \cdot \gamma_2 \cdot \alpha^{-1}]|\text{Im}(\gamma_i) \subset U_i, U_i \in \mathcal{U}_n, i = 1, 2 \rangle.
\]

Note $\pi^{sp}(\mathcal{U}_n, x_0) \subseteq \Pi^{sp}(\mathcal{U}_n, x_0)$

$\Pi^{sp}(\mathcal{U}_m, x_0) \subseteq \pi^{sp}(\mathcal{U}_n, x_0)$ for large enough $m = m(n) \geq n$ by paracompactness.

**Remark:** $\pi^{sp}(X, x_0) = \bigcap_{n \geq 1} \Pi^{sp}(\mathcal{U}_n, x_0)$
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**Thick Spanier groups**

**Definition:** The thick Spanier group of $X$ with respect to $\mathcal{U}_n$ is the normal subgroup

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**Remark:** $\pi^{sp}(X, x_0) = \bigcap_{n \geq 1} \Pi^{sp}(\mathcal{U}_n, x_0)$
The fundamental group of a Peano continuum
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$\Pi^{sp}(\mathcal{U}_m, x_0) \subseteq \pi^{sp}(\mathcal{U}_n, x_0)$ for large enough $m = m(n) \geq n$ by paracompactness

Remark: $\pi^{sp}(X, x_0) = \bigcap_{n \geq 1} \Pi^{sp}(\mathcal{U}_n, x_0)$
**Theorem (B, Fabel):** There is a level short exact sequence

\[
1 \longrightarrow \Pi^{sp}(U_n, x_0) \longrightarrow \pi_1(X, x_0) \overset{(p_n)_*}{\longrightarrow} \pi_1(X_n, x_n) \longrightarrow 1
\]

Applying $\lim\limits_{\leftarrow n}$ we obtain

\[
1 \longrightarrow \pi^{sp}(X, x_0) \longrightarrow \pi_1(X, x_0) \overset{\psi}{\longrightarrow} \check{\pi}_1(X, x_0)
\]

In particular,

\[
\ker \psi = \pi^{sp}(X, x_0),
\]

\[
\check{\pi}_1(X, x_0) = \lim_{\text{regular } p} \text{coker}(p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)).
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\]
Comparison

Lemma: Each of the collections
1. \( \pi_{\text{sp}}(U_n, x_0) \) for \( n \geq 1 \),
2. \( \Pi_{\text{sp}}(U_n, x_0) \) for \( n \geq 1 \),
3. \( N_{\pi_{\text{qtop}}^1}(X, x_0) \) for \( N_{\text{open}} \)

is cofinal in the other two (when directed by inclusion).

Theorem: If \( X \) is a Peano continuum, then
\[ \ker \Psi = \pi_{\text{sp}}(X, x_0) = \bigcap N_{\pi_{\text{qtop}}^1}(X, x_0) \]

Corollary: If \( X \) is a Peano continuum, then \( X \) is \( \pi_1 \)-shape injective \( \iff \pi_{\text{qtop}}^1(X, x_0) \) is invariantly separated.
Comparison

**Lemma:** Each of the collections

1. $\{\pi^{sp}(U_n, x_0) | n \geq 1\}$,
2. $\{\Pi^{sp}(U_n, x_0) | n \geq 1\}$,
3. $\{N \leq \pi^{qtop}_1(X, x_0) | N \text{ open}\}$

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**Theorem:** If $X$ is a Peano continuum, then

$$\ker \Psi = \pi^{sp}(X, x_0) = \bigcap_{N \leq \pi^{qtop}_1(X, x_0) \text{ open}} N.$$

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**Lemma:** Each of the collections

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**Theorem:** If \( X \) is a Peano continuum, then

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\ker \psi = \pi^{sp}(X, x_0) = \bigcap_{N \leq \pi^{qtop}_1(X, x_0) \text{ open}} N.
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**Corollary:** If \( X \) is a Peano continuum, then \( X \) is \( \pi_1 \)-shape injective \( \iff \pi^{qtop}_1(X, x_0) \) is invariants separated.
The data of the fundamental group of a Peano continuum $X$ retain by each of

1. the covering spaces of $X$,
2. the shape of $X$,
3. open normal subgroups of $\pi_1^{qtop}(X, x_0)$.

is precisely the same.
Conclusion

The data of the fundamental group of a Peano continuum $X$ retain by each of

1. the covering spaces of $X$,
2. the shape of $X$,
3. open normal subgroups of $\pi_1^{qtop}(X, x_0)$.

is precisely the same.

1. and 2. are exhausted but the topology of $\pi_1^{qtop}(X, x_0)$ is rarely generated by open normal subgroups.
Other data retained by $\pi_{1}^{qtop}(X, x_0)$
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<table>
<thead>
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<th>Interpretation</th>
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<td>$\pi_1$-shape injective</td>
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<tr>
<td>Totally separated</td>
<td>$\Omega(X, x_0)$ is $\pi_0$-shape injective</td>
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Other data retained by $\pi_{1}^{qtop}(X, x_0)$

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Example in cylindrical coordinates

The topology of $\pi_1^{qtop}(X, x_0)$ can topologically distinguish homotopy classes which are indistinguishable using shape/coverings.

Example (Conner, Meilstrup, Repovš, Zastrow, Željko):

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5. $\mathcal{S} = C \cup S \cup \bigcup_{d \in D} A_d$ is a Peano continuum such that $\ker \Psi \neq 1$ but $\pi_1^{qtop}(X, x_0)$ is $T_1$ (Fischer, Repovš, Virk, Zastrow) & (B, Fabel)
Open problems

**Problem 1:** If $X$ is a Peano continuum and $\pi_1^{qtop}(X, x_0)$ is $T_2$, must $\pi_1^{qtop}(X, x_0)$ be invariantly separated (i.e. $X$ $\pi_1$-shape injective)?

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Thank you!