Shilnikov saddle-node bifurcation

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Saddle-saddle or Shilnikov saddle-node bifurcation gives rise to complex dynamics in a system after a merger of two saddles connected globally by \( m \geq 2 \) heteroclinic orbits. The later ones become the transverse homoclinic connections of the saddle-saddle at the bifurcation, as illustrated in Figure 1. After the saddle-saddle vanishes through this codimension-one bifurcation, the system’s dynamics becomes conjugate to a suspension over a Bernoulli scheme on same \( m \) symbols.

Single zero exponent

Let us consider an \( n \)-dimensional, sufficiently smooth system that has a structurally unstable equilibrium state \( O \) at the origin with eigenvalues \( \text{Re} \lambda_i \neq 0 \) (\( i = 1, \ldots, n-1 \)), and \( \text{Re} \lambda_n = 0 \). Near the origin, the system can then be written in the form [L. Shilnikov et al., 1998 and 2001]:

\[
\dot{x} = f(x, y), \quad \dot{y} = Ay + g(x, y),
\]

where \( f, g \) vanish at the origin along with their first partials. It is supposed that in the Taylor expansion

\[
f(x, \varphi(x)) = l_2x^2 + l_3x^3 \ldots,
\]

the first Lyapunov coefficient \( l_2 \neq 0 \); here \( y = \varphi(x) \) is the solution of the equation \( Ay + g(x, y) = 0 \); here \( A \) is an \( (n-1) \times (n-1) \) matrix.

The behavior of the solutions of the system (1) near the origin is similar to that of the following truncated normal form

\[
\dot{x} = \mu + x^2, \quad \dot{y} = Ay.
\]

Depending on the spectrum of the matrix \( A \), the following cases are possible:

1) The case \( \text{Re} \lambda_i < 0 \), \( i = 1, \ldots, n-1 \), is illustrated in Figure 2.

Here, the \( (n-1) \)-dimensional non-leading or strongly stable manifold \( W^{ss} \) breaks a neighborhood of the equilibrium state into the two regions: stable (or node) and saddle. The saddle one contains the trajectories \( \Gamma^s \) that originate from \( O \), or converge to it as \( t \to -\infty \), while in the stable node region the trajectories converge to the origin in the forward time \( t \to +\infty \).
Under small, smooth perturbations of the vector field, \( O \) either vanishes or decouples into two equilibrium states: a stable node and a saddle. This is the local saddle-node bifurcation.

2) The case \( \text{Re} \lambda_i > 0, i = 1, \ldots, n-1 \), is reduced to the previous one by the time change \( t \rightarrow -t \), which makes the unstable node stable, while preserving the saddle structure of the other equilibrium state.

3) The case \( \text{Re} \lambda_i > 0, i = 1, \ldots, k \), and \( \text{Re} \lambda_j > 0, j = k+1, \ldots, n-1 \). The corresponding equilibrium state is called a saddle-saddle, as it is the result of a merger of two saddles of the proper topological types, see Figure 1 and Figure 5. It has a \((k+1)\)-dimensional stable manifold \( W^s \), which is diffeomorphic to a half-space \( \{R x_1, \ldots, x_k, x < 0\} \), as well as an unstable one \( W^u \) diffeomorphic to a half-space \( \{R x_1, \ldots, x_{n-k-1}, x > 0\} \).

### Saddle-node bifurcation for synchronization

Back in the early 30's of the last century, A. Andronov and A. Vitt studied the phenomena of the 1:1 resonance in the periodically forced Van der Pol equation

\[
\ddot{x} - \mu(1-x^2)\dot{x} + \omega_0^2 x = A \sin \omega t,
\]

where \( \mu \ll 1 \) and \( |\omega_0 - \omega| \sim \mu \). They had discovered that the disappearance of the stable equilibrium state was accompanied by the emergence of a stable limit cycle in the system: or in other terms, they discovered the transition mechanism between synchronization (phase locking) and modulation (beating oscillations). A rigorous mathematical explanation of this phenomenon was later given by A. Andronov and E. Leontovich with the use of the tools of global bifurcations.

**Theorem 1.** Let a two-dimensional system have an equilibrium state \( O \) of the saddle-node type with the characteristic exponents \( \lambda_1 < 0 \) and \( \lambda_2 = 0 \) such that its unstable separatrix \( \Gamma^u \) comes back to \( O \) as \( t \rightarrow +\infty \) not in \( W^{ss} \). Then, as the saddle-node has vanished, a single, stable periodic orbit emerges from its homoclinic loop \( \Gamma^u \).

The generalization of this bifurcation for a high-dimensional case was done by L. Shilnikov [1963]. Its stages are sketched in Figure 3.

### Saddle-saddles
In addition to the saddle-node, L. Shilnikov proposed and examined another peculiar homoclinic bifurcation of a saddle-saddle in [1969]:

**Theorem 2.** Let \( \Gamma \in W^s \cap W^u \) and not in \( \partial W^s \cap \partial W^u \), i.e., \( W^u \) and \( W^s \) cross along \( \Gamma \) transversally. Then, after the saddle-saddle has vanished, \( \bar{\Gamma} \) becomes a single, saddle periodic orbit (Figure 4).

The proof of this theorem called for a creation of the special technique of solving the boundary value problem known, today as the Shilnikov coordinates, which is especially helpful for proving the existence of complex hyperbolic dynamics generated by saddle orbits.

**Theorem 3.** Let \( W^u \) and \( W^s \) of a saddle-saddle cross transversally along \( \Gamma_1, \cdots, \Gamma_m \) within some neighborhood \( U \) (see Figure 1). Then, after the saddle-saddle has disappeared, the hyperbolic set conjugate to a suspension over a subshift on \( m \) symbols is born in \( U \).

In general, in a codimension-one case, \( W^u \) and \( W^s \) of a saddle-saddle may intersect transversally over a countable set of such homoclinic orbits; i.e. the closure of this set includes countably many saddle periodic orbits. If so, the disappearance of the saddle-saddle gives rise to the emergence of a one-dimensional hyperbolic set, to describe which one needs to employ topological Markov chains with finite numbers of states. [Afraimovich and Shilnikov, 1983]

### Applications and Examples

The complex set resulting from the Shilnikov saddle-saddle bifurcation set is not an attractor, in general. That is perhaps the reason why a realistic application of this simple co-dimension-1 bifurcation is still wanted. An example was proposed by Glendenning and Sparrow [1996]: their underlying idea was to get rid of the saddle point at the origin in a Lorenz-like model through the saddle-saddle bifurcation, right when the bifurcating equilibrium state has a homoclinic butterfly formed simultaneously by two homoclinic loops, like in Figure 6. This would have made the strange attractor of the system truly hyperbolic [Afraimovich et al, [1977, 1983]], if its orbits did not get away from it down along the z-axis. It follows from these references as well that since the pre-images of the stable manifold of the saddle at the origin are dense everywhere in the attractor, the phase point shall inevitably pass arbitrarily close by the z-axis again even after the equilibrium state is gone. It is dragged down in z to be re-injected back (Figure 7) so that while lowering down to the strange attractor from above it is turning around the z-axis. This leads to the formation of the distinctive hooks [Shilnikov, [1993]] in the 2D Poincare mapping, thereby indicating that the the necessary property of tranversality in crossings of the stable and unstable foliations in the Lorenz attractor is no longer persisted.

### References


**Internal references**


**See Also**

Bifurcations, Shilnikov bifurcation, Homoclinic Bifurcations, Homoclinic Orbits, Lorenz attractor, Smale Horseshoe, Saddle-focus, Saddle-node, Chaos

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Accepted on: 2008-04-23 04:22:19 GMT

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