Problem: Suppose you have twelve golden coins, all the same in shape and size. However, one of them is fake: it does seem golden from the outside, but its core is made of a cheaper material, so its weight differs from the weight of others (all the others weigh equally). Using the scales as in the picture below, and taking at most three measurements, how can you find which coin is fake AND whether it is lighter or heavier than the others? You need to describe a procedure/algorithim so that whatever happens in each of the three measurements, the procedure will in any case lead to the answer.

Please submit your solution to:

- Dr. Marko Samara, msamara@gsu.edu

before the deadline: February 28, 7:00PM. The WINNER will be awarded with a $15 dining gift card and certificate, and will be announced in the next issue of the Problem of the Month.

Student with correct answer to the January’s problem: Ruoyi Chen (Congratulations!)
Solution to the January’s Problem of the Month

Problem: For each \( n \in \mathbb{N} \), how many triangles in total do you see?
In the following sequence of pictures, each construction consists of small equilateral triangles, all of the sides of length, say, \( a \).

If \( n \) is any natural number, let \( S_n \) be the total number of triangles in the \( n \)-th step. For example, in the step \( n = 1 \), you see only one triangle (which is of side \( a \)), and so, \( S_1 = 1 \). In the step \( n = 2 \), there are four triangles of side \( a \), and one of side 2\( a \), and so, \( S_2 = 4 + 1 = 5 \). For \( n = 3 \), we have \( S_3 = 9 + 3 + 1 = 13 \). Similarly, \( S_4 = 16 + 7 + 3 + 1 = 27 \) triangles (note that one of the 7 triangles of side 2\( a \) is upside-down).

For any \( n \in \mathbb{N} \), find the formula for \( S_n \) in terms of \( n \). (Don’t forget to include all upside-down triangles. You may want to draw couple of more steps.)

Solution:

\[
S_n = \begin{cases} 
\frac{n(n+2)(2n+1)}{8}, & \text{if } n \text{ is even} \\
\frac{n(n+2)(2n+1) - 1}{8}, & \text{if } n \text{ is odd}
\end{cases}
\]

Proof:
Let \( n \in \mathbb{N} \). Denote by \( T_n \) and \( U_n \) total number of "upright" and "upside-down" triangles, respectively. So, \( S_n = T_n + U_n \). Let us draw two more pictures, for steps \( n = 5 \) and \( n = 6 \).

To find \( T_n \), note that there are 1 + 2 + ... + \( n \) upright triangles of side \( a \), 1 + 2 + ... + (\( n - 1 \)) upright triangles of side 2\( a \), 1 + 2 + ... + (\( n - 2 \)) upright triangles of side 3\( a \), and so on. The largest possible upright triangle is of side \( na \) and there is only one of them. Summing up these values, and using the formulæ

\[
\sum_{i=1}^{m} i = \frac{m(m+1)}{2}, \quad \sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}
\]

for all \( m \in \mathbb{N} \), we get

\[
T_n = \sum_{k=1}^{n} k + \sum_{k=1}^{n-1} k + \ldots + \sum_{k=1}^{2} k + \sum_{k=1}^{1} k = \sum_{j=1}^{n} \sum_{k=1}^{j} k = \sum_{j=1}^{n} \frac{j(j+1)}{2}
\]

\[
= \frac{1}{2} \left( \sum_{j=1}^{n} j^2 + \sum_{j=1}^{m} j \right) = \frac{1}{2} \left( \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right) = \frac{n(n+1)(n+2)}{6}.
\]
As for $U_n$, it’s a bit trickier.
First not that upside-down triangle of side $a$ appears for the first time in the step $n = 2$
and in each following step their number increases by increment which is by one larger than
the increment in the previous step. For example, in step $n = 3$ there are $1+2$ such tri-
angles, in step $n = 4$ there are $1+2+3$ such triangles and so on. So, in step $n$, there are
$1 + 2 + ... + (n - 1)$ of these triangles.
Next, note that first triangle of side $2a$ appears in the 4th step, then in step 5 there are
$1+2$ such triangles, in step 6 there are $1+2+3$ of them and so on. In step $n$, there are
$1 + 2 + ... + (n - 3)$ such triangles.
Also, the first triangle of side $3a$ appears in the 6th step, and in the step $n$ there are
$1 + 2 + ... + (n - 5)$.
Continuing with this observation, we see that for $1 \leq j \leq n/2$, the first triangle of side $ja$
appears in the $(2j)$-th step and in step $n$ there are $1 + 2 + ... + (n - 2j + 1)$ such triangles.
Also, the largest triangle is of side $\lfloor n/2 \rfloor \cdot a$ (where $\lfloor n/2 \rfloor$ means the greatest integer of $n/2$, 
i.e. the largest integer not exceeding $n/2$). Furthermore, if $n$ is even, the largest triangle
(which is of side $\frac{n}{2}a$), appears only once in the $n$-th step. On the other hand, if $n$ is odd, the
largest triangle (which is of side $\lfloor n/2 \rfloor = (n - 1)/2$) appears $1 + 2 = 3$ times in the $n$-th step.
Summarizing these observations, we get

$$U_n = \begin{cases} 
\sum_{k=1}^{n-1} k + \sum_{k=1}^{n-3} k + \ldots + \sum_{k=1}^{1} k, & \text{if } n \text{ is even} \\
\sum_{k=1}^{n-1} k + \sum_{k=1}^{n-3} k + \ldots + \sum_{k=1}^{n/2} k, & \text{if } n \text{ is odd} 
\end{cases} = \begin{cases} 
\sum_{j=1}^{n/2} \sum_{k=1}^{2j-1} k, & \text{if } n \text{ is even} \\
\sum_{j=1}^{(n-1)/2} \sum_{k=1}^{2j} k, & \text{if } n \text{ is odd} 
\end{cases}$$

$$= \begin{cases} 
\sum_{j=1}^{n/2} \frac{(2j - 1) \cdot 2j}{2}, & \text{if } n \text{ is even} \\
\sum_{j=1}^{(n-1)/2} \frac{2j \cdot (2j + 1)}{2}, & \text{if } n \text{ is odd} 
\end{cases} = \begin{cases} 
2 \cdot \sum_{j=1}^{n/2} j^2 - \sum_{j=1}^{n/2} j, & \text{if } n \text{ is even} \\
2 \cdot \sum_{j=1}^{(n-1)/2} j^2 + \sum_{j=1}^{(n-1)/2} j, & \text{if } n \text{ is odd} 
\end{cases}$$

$$= \begin{cases} 
2 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \cdot \frac{(n+1)}{2}, & \text{if } n \text{ is even} \\
2 \cdot \frac{(n-1)(n+1) + (2, n-1 + 1)}{6} - \frac{n-1}{2} \cdot \frac{(n+1)}{2}, & \text{if } n \text{ is odd} 
\end{cases} = \begin{cases} 
\frac{n(n+1)(2n-1)}{24}, & \text{if } n \text{ is even} \\
\frac{(n-1)(n+1)(2n-3)}{24}, & \text{if } n \text{ is odd} 
\end{cases}$$

Adding up $T_n$ and $U_n$, we finally get

$$S_n = T_n + U_n = \frac{n(n+1)(n+2)}{6} + \begin{cases} 
\frac{n(n+1)(2n-1)}{24}, & \text{if } n \text{ is even} \\
\frac{n-1)(n+1)(2n-3)}{24}, & \text{if } n \text{ is odd} 
\end{cases} = \begin{cases} 
\frac{n(n+1)(2n+1)}{8}, & \text{if } n \text{ is even} \\
\frac{n+2)(2n+1)}{8} - 1, & \text{if } n \text{ is odd} 
\end{cases}$$

In light of comparing even and odd formula, one can expand the two numerators and see
that the numerator of the odd formula is by 1 less than that of the even formula. Thus, we
can also write

$$S_n = \begin{cases} 
n(n+2)(2n+1), & \text{if } n \text{ is even} \\
n(n+2)(2n+1) - 1, & \text{if } n \text{ is odd} 
\end{cases}$$