Sample problems for the Qualifying Examination in Algebra at GSU - Prepared by F. Enescu and Y. Yao

These problems represent practice problems aiming to indicate the level of difficulty of the problems on the exam. The students are encouraged to cover all the topics listed in the syllabus and not limit themselves to the list below.

(1) Show that in the ring $\mathbf{Z}[\sqrt{-3}]$, $1+\sqrt{-3}$ and $1-\sqrt{-3}$ are relatively prime.
(2) Find the g.c.d., if it exists, of $10+11i$ and $8+i$ in $\mathbf{Z}[i]$.
(3) Let R be a ring with identity and S the ring of all $n \times n$ matrices with entries in R. Prove that J is a two-sided ideal of S if and only if J equals the set of $n \times n$ matrices with entries in a two-sided ideal I of R.
(4) Let R be a commutative ring (with identity). Show that an ideal I of R is maximal if and only if for every $r \notin I$ there exists $x \in R$ such that $1 - rx \in I$.
(5) Let R be a ring. Show that $f \in R[x]$ is a zero divisor if and only if there exists $b \in R$, nonzero, such that $b \cdot f = 0$.
(6) Let R be a PID. Show that any proper ideal is product of maximal ideal, uniquely determined up to order.
(7) Let F be a field. Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)]$.
(8) Let F be a field and let $F(X)$ be the field of rational functions over F. Let L be a subfield of $F(X)$ containing strictly F. Show that $L \subseteq E$ is algebraic.
(9) Let E be the splitting field over F of a polynomial of degree n. Prove that $[E : F] \le n!$.
(10) For any prime p and $a \in \mathbf{F}_p$, $a \neq 0$, let $f(X) = X^p - X + a$.
Find $Gal(K/\mathbf{F}_p)$ where K is the splitting field of f.
(11) Consider the field $\mathbf{Q}(X)$ and its automorphism σ that sends

$$f(X) \to f(X+1)$$

for all rational functions $f(X)$. Show that σ generates an infinite subgroup of $Aut(\mathbf{Q}(X))$. Determine the fixed field of this subgroup.
(12) Let $\mathbf{Q}(X)$ the field of rational functions in X over $\mathbf{Q}$. Let $a = \frac{X^3}{X+1}$. What is $[\mathbf{Q}(X) : \mathbf{Q}(a)]$? Justify your answer.
(13) Let K be the splitting field of $X^3 - X - 1$ over $\mathbf{Q}$. Determine all the subfields of K containing $\mathbf{Q}$.
(14) Let $\alpha = \sqrt{1+\sqrt{2}}$. Describe the Galois group of $\mathbf{Q}(\alpha)$ over $\mathbf{Q}$. Give an example of a Galois extension of $\mathbf{Q}$ strictly contained in $\mathbf{Q}(\alpha)$.
(15) (5 points) Let $F = \mathbf{Q}(\sqrt{2})$, $E = \mathbf{Q}(\sqrt[4]{2})$. Show that E is a normal extension of F, F is a normal extension of $\mathbf{Q}$. but E is not a normal extension of $\mathbf{Q}$.
(16) (5 points) Let $f(x)$ be a polynomial of degree n over a field F of characteristic p. Suppose $f'(x) = 0$. show that $p \mid n$ and $f(x)$ has at most n/p distinct roots.
(17) Compute the Galois group of $X^5 - 1$ over $\mathbf{Q}$.
(18) Let F be a finite field of characteristic $p > 0$ and $f \in F[X]$ be an irreducible polynomial. Show that that there exists n such that f divides $X^{p^n} - X$ over $F[X]$.
(19) Let E be the splitting field of $X^4 + 1$ over $\mathbf{Q}$. How many intermediate fields $\mathbf{Q} \subseteq K \subseteq E$ are normal extensions of $\mathbf{Q}$? Please explain your answer fully.
(20) Compute the Galois group of $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbf{Q}$.
(21) Prove that $\mathbf{F}_p(x^p, y^p) \subseteq \mathbf{F}_p(x, y)$ is not a simple extension.

(22) Show there is no simple group of order 12.
(23) Show there is no simple group of order p^2q, where p and q are distinct primes.
(24) Show there is no simple group of order 30.
(25) Show there is no simple group of order 36.
(26) Show every group of order pq, where $p < q$ are prime numbers such that $q \not\equiv 1 \mod p$, is cyclic.
(27) If G is a group of order $2p^n$ with p a prime and $n > 0$, show G is not simple.
(28) Let G be a group with $2 \le |G| < 60$. Show that G is simple if and only if $|G|$ is prime.
(29) If G is a simple group of order 60, show G is isomorphic to A_5.
(30) Let G be a finite group and P is a Sylow p-subgroup of G where p is a prime divisor of $|G|$. Show $N(N(P)) = N(P)$, where $N(-)$ denotes the normalizer.
(31) Let G be a group, $H < G$ such that $|H| < \infty$, and P be a Sylow p-subgroup of H, in which p is a prime divisor of $|H|$. Show $N(H) \subseteq HN(P)$.
(32) Let G be a finite abelian group. For any positive integer m that divides $|G|$, show there exists $H < G$ such that $|H| = m$.
(33) Let G be a group. Show that $|Aut(G)| = 1$ if and only if $|G| \le 2$. (You may assume $|G| < \infty$. But better if you can prove it without the assumption.)
(34) Let G be a finite group and $H < G$. If $H \ne G$, show $\cup_{g \in G} gHg^{-1} \ne G$.
(35) Let G be a p-group with p a prime number and X a finite G-set. If p does not divide $|X|$, show there is a singleton orbit.
(36) Let M and N_i with $i \in \Lambda$ be modules over a ring R. If M is a finitely generated R-module, show $Hom_R(M, \oplus_{i\in\Lambda} N_i) \cong \oplus_{i\in\Lambda} Hom_R(M, N_i)$.
(37) Let R be a PID and M be a free R-module. Show that every finitely generated R-submodule of M is free over R.
(38) Let R be a PID and M be a free R-module of finite rank. Show that every R-submodule of M is free over R.
(39) Let R be a PID and M be a finitely generated R-module. Show that $M/T(M)$ is free over R, in which $T(M)$ is the torsion submodule of M.
(40) Let M be the quotient of $\mathbb{Z}^4$ modulo the submodule generated by the column vectors of the following 4×5 matrix

$$A = \begin{pmatrix} 12 & -12 & 12 & 24 & 24 \\ -12 & 12 & 24 & -24 & 12 \\ 24 & -12 & 24 & 48 & 60 \\ 0 & 12 & 36 & 0 & 48 \end{pmatrix}$$

Find the Smith normal form of A. Then express M as a direct sum of cyclic $\mathbb{Z}$-modules. What are the free rank, the invariant factors, and the elementary divisors of M?
(41) Let $R = \mathbb{Q}[x]$ and M be the quotient of R^3 modulo the submodule generated by the column vectors of the matrix $xI - A$ where

$$A = \begin{pmatrix} -7 & -10 & -1 \\ 7 & 10 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$

Find the Smith normal form of $xI - A$ and express M as a direct sum of cyclic R-modules. What are the free rank, the invariant factors and the elementary divisors of M?
(42) Let R be a PID and M be a free R-module of finite rank. For any finitely generated R-module N, show $Hom_R(N, M)$ is free over R. (You may assume the fact that $Hom_R(N, M)$ is finitely generated over R. Better if you can prove this fact.)
(43) Let R be a commutative ring with 1 and $f(x) \in R[x]$ be a monic polynomial. Show $R[x]/(f(x))$ is a free R-module.
(44) Let $f(x) \in \mathbb{Z}[x]$ and $I = (f(x))$ be the ideal of $\mathbb{Z}[x]$ generated by $f(x)$. Show there is a ring homomorphism from $\mathbb{Z}[x]/I$ to $\mathbb{Z}$ sending $\bar{1}$ to 1 if and only if $f(x)$ has an integer root.