(1) Let $\mathbb{Q}$ be the field of all rational numbers, $R$ the ring of all $4 \times 4$ matrices with entries in $\mathbb{Q}$, and $J$ a two-sided ideal of $R$ such that $A \in J$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Determine whether the $4 \times 4$ identity matrix $I_4$ is contained in $J$. Justify your claim fully.
(b) Determine whether $J$ coincides with $R$. Explain why.

(2) Let $R$ be a commutative ring (with identity). Assume that $R$ has only three distinct ideals: 0, $I$, $R$. Prove that
(a) If $a \in I$ then $1 - a$ is invertible in $R$.
(b) If $a, b$ are nonzero elements in $I$ then $ab = 0$.

(3) Show there is no simple group of order 72.

(4) Let $R = \mathbb{Q}[x]$ and $M$ be the quotient of $R^3$ modulo the $R$-submodule generated by the columns of the $3 \times 3$ matrix $xI - A$ where $I$ is the $3 \times 3$ identity matrix and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix}.$$

(a) Find the Smith normal form of $xI - A$.
(b) Up to isomorphism, express $M$ as a direct sum of cyclic $R$-modules.
(c) What are the free rank, the invariant factors, and the elementary divisors of $M$?
(d) Determine the rational canonical form and the Jordan canonical form of $A$.

(5) Compute the Galois group of $\mathbb{Q}(\sqrt{5}, \sqrt{3})$ over $\mathbb{Q}$.

(6) Prove that $\mathbb{F}_p(x^p, y^p) \subseteq \mathbb{F}_p(x, y)$ is not a simple extension. (Here $p$ is a prime number.)