Calculus Challenge 2003

1) Which is larger (for $n > 8$): $(\sqrt{n + 1})^{\sqrt{n}}$ or $(\sqrt{n})^{\sqrt{n} + 1}$?
2) a) Show that \( \lim_{n \to \infty} \sqrt{n^{200} + n^{100} + 1} - n^{100} = 1/2 \).

b) Compute \( \lim_{n \to \infty} \sin^2(\pi \sqrt{n^{200} + n^{100} + 1}) \).
3) Define a sequence by:

\[ a_n = \int_0^1 (1 - x^2)^n \, dx. \]

a) Show that \( \lim_{n \to \infty} \sqrt[n]{a_n} = 1. \)

b) Calculate \( \sum_{n=1}^{\infty} a_n. \)
4) a) Show that $\sqrt{\alpha} \leq \frac{1+\alpha}{2}$ for all positive numbers $\alpha$.

b) Show that the sequence $x_n$ converges, where

$$x_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}}.$$
5) Let $f$ be a continuous function such that $f$ is strictly increasing, $f(0) = 0$ and $f(1) = 1$. Let $g$ be the inverse of $f$ (so that for every $x$, $f(g(x)) = x$ and $g(f(x)) = x$). Show that

$$\int_0^1 f(x) \, dx + \int_0^1 g(y) \, dy = 1.$$
6) Suppose \( \lim_{x \to \infty} (f(x) + f'(x)) = 0 \), and that \( \lim_{x \to \infty} f(x) \) exists. Show that

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} f'(x) = 0.
\]