Lecture 12

1. Integral extensions and the Going-Up Theorem

Definition 1.1. Let \( \phi : R \to S \) be a ring homomorphism. Then \( \phi^* : \text{Spec}(S) \to \text{Spec}(R) \) by mapping each \( Q \) to \( Q \cap R = \phi^{-1}(Q) \). Let \( P \in \text{Spec}(R) \). Then a prime ideal \( Q \in \text{Spec}(S) \) lies over \( P \) if \( Q \cap R = P \), i.e., \( \phi^*(Q) = P \).

Proposition 1.2. If \( R \subseteq S \) is an integral extension of rings, \( I \leq R \), and \( u \in IS \cap R \), then \( \exists n \) such that \( u^n \in I \). Moreover, if \( I = \text{Rad}(I) \) (in particular if \( I \) is prime), then \( IS \cap R = I \).

Proof. Let \( u \in IS \). Then \( u = \sum_{t=1}^{n} j_t \theta_t \), with \( \theta_t \in S \) and \( j_t \in I \). So we may assume that \( S = R[\theta_1, \ldots, \theta_n] \), (i.e., \( S \) is integral and finitely generated over \( R \)). It suffices to show that if \( u \in IR[\theta_1, \ldots, \theta_n] \cap R \), then \( u \in \text{Rad}(I) \). Since \( S \) is module finite over \( R \), we can write \( S = Rs_1 + \cdots + Rs_m \) by taking \( s_1 = 1 \) if \( 1 \notin \{s_1, \ldots, s_n\} \). For each \( k \), \( us_k = \sum_{j=1}^{m} \alpha_{kj} s_j \) with \( \alpha_{kj} \in I \), since \( uS \subseteq IS \). So the matrix \( [uI - \alpha_{kj}] \) times the column vector of \( s \)'s equals to zero. But \( s_1 = 1 \) and the determinant of that matrix must be zero. So, by expanding the determinant, \( u^n + r_{m-1}u^{m-1} + \cdots + r_1 u + r_0 = 0 \), with \( r_i \in I^{m-i} \), and all but the first term in \( I \). This implies \( u^n \in I \).

The second part of the statement follows immediately from the first.

\[ \square \]

Proposition 1.3. Let \( h : R \to S \) be a homomorphism of rings, and let \( P \in \text{Spec}(R) \). Then the following assertions are equivalent:

(i) \( \exists Q \in \text{Spec}(S) \) with \( h^{-1}(Q) = Q \cap R = P \);

(ii) \( \text{Im}(R \setminus P) \cap PS = \emptyset \);

(iii) \( h^{-1}(PS) = P \).

Proof. The equivalence between (ii) and (iii) is clear by definition of \( h \). For (i) implies (ii), if \( \exists Q \) with \( Q \in \text{Spec}(S) \) and \( Q \cap R = P \), then \( P \subseteq h^{-1}(PS) \subseteq h^{-1}(Q) = Q \cap R = P \). So \( P = h^{-1}(PS) \). For (ii) implies (i), note that \( R \setminus P \) is a multiplicative set in \( R \), so
Proposition 1.4. Let $R \subseteq S$, with $S$ a domain, and take $0 \neq s \in S$, with $s$ integral over $R$. Then there is a nonzero multiple of $s$ in $R$. Moreover, if $0 \neq J \subseteq S$, then $J \cap R \neq 0$.

Proof. Since $s$ is integral over $R$, $\exists n$ such that $s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0$ with $a_i \in R$. Let $n$ be minimal among all integral dependence equations. Then $a_0 \neq 0$ because if $a_0 = 0$ then $s(s^{n-1} + \cdots + a_1) = 0$ and since $s \neq 0$, it gives that $s^{n-1} + \cdots + a_1 = 0$. Since $S$ is a domain, but this contradicts $n$ being minimal. So $s(s^{n-1} + \cdots + a_1) = -a_0 \in R \setminus \{0\}$.

For the second part of the statement, let $0 \neq s \in J$. Then a power of $s$ is in $R$, hence in $J \cap R$, and it is of course nonzero since $R$ is a domain.

Our next goal is to prove the following theorem, but we will need to develop a few ideas along the way.

Theorem 1.5 (Going-up Theorem). Let $R \subseteq S$ be an extension of rings, and assume that $S$ is integral over $R$. Then the following are true:

1. (Lying Over) For every $P \in \text{Spec}(R)$, $\exists Q \in \text{Spec}(S)$ that lies over $P$, i.e., $Q \cap R = P$.
2. (Incomparability) If $Q \neq Q'$, with $Q, Q' \in \text{Spec}(S)$, and $Q \cap R = Q' \cap R = P$, then $Q$ and $Q'$ are incomparable, i.e., if $Q \subset Q'$ then $Q \cap R \subset Q' \cap R$.
3. (Going-Up) If $P_0 \subset P_1 \subset \cdots \subset P_n$

is a strictly ascending chain of prime ideals in $R$, then there is a strictly ascending chain of prime ideals in $S$,

$Q_0 \subset Q_1 \subset \cdots \subset Q_n$

with $Q_i \cap R = P_i$ for every $0 \leq i \leq n$.

Proof. (1) According to Proposition 1.3, we need to show that $PS \cap R = P$. But this follows at once from Proposition 1.2.
(2) Let us assume that there exists a pair of prime ideals $Q \subsetneq Q'$ such that $Q \cap R = Q' \cap R = P$.

Note that $R \hookrightarrow S$ induces a natural map $R/P \hookrightarrow S/Q$ that takes $\overline{a} \mapsto \overline{a}$. Now, since $Q \cap R = P$, the above map is injective, that is it gives an extension of rings. But $R \hookrightarrow S$ is integral, so our new map is also integral. Moreover, $S/Q$ is a domain. We can apply Proposition 1.4 to conclude $\frac{Q'}{Q} \cap \frac{R}{P} \neq 0$. But $Q' \cap R = P$ gives that $\frac{Q'}{Q} \cap \frac{R}{P} = 0$. So we obtained a contradiction.

(3) It suffices to show the claim for $n = 1$, so given $P \subseteq P'$ and $Q \cap R = P$, find $Q'$. We have $R/P \hookrightarrow S/Q$ an integral extension of domains, and $P'/P \in \text{Spec}(R/P)$. Applying part (1), we know that there exist a prime ideal in $S/Q$ say $\frac{Q'}{Q}$ such that $\frac{Q'}{Q} \cap \frac{R}{P} = P'$, so $Q' \cap R = P'$, with $Q' \in \text{Spec}(S)$.

\[\square\]

**Corollary 1.6.** If $R \hookrightarrow S$ is an integral extension, then $\dim R = \dim S$.

*Proof.* Given a chain of primes in $S$, by intersecting each term with $R$, we obtain a chain of primes in $R$. Applying the part (2) of the above theorem we get $\dim S \leq \dim R$. Parts (1) and (3) show $\dim R \leq \dim S$. \[\square\]

**Corollary 1.7.** If $R \hookrightarrow S$ is an integral extension and $u \in R$ has an inverse in $S$, then $u$ has an inverse in $R$.

*Proof.* We know $uR$ is an ideal in $R$. If $u$ has no inverse in $R$, then $uR \subseteq \mathfrak{m}$, where $\mathfrak{m}$ is maximal in $R$. By the lying over property, $\exists Q \in \text{Spec}(S)$ such that $uS \subseteq Q \subset S$, a contradiction, since $uS = S$. \[\square\]

**Corollary 1.8.** Let $R \hookrightarrow S$ be an integral extension. If $S$ is a field then $R$ is a field.

*Proof.* Follows at once from the above Corollary. \[\square\]

**Corollary 1.9.** Let $R \hookrightarrow S$ be an integral extension. Then if $M \in \text{Max}(S)$, then $M \cap R = \mathfrak{m} \in \text{Max}(R)$.

*Proof.* Since $M \cap R \in \text{Spec}(R)$, then $R/\mathfrak{m} \hookrightarrow S/M$ is an integral extension of domains, and so $S/M$ is a field. This implies $\mathfrak{m} \in \text{Max}(R)$. \[\square\]
2. The dimension of a polynomial ring over a field

**Definition 2.1.** Let $K \subseteq L$ a field extension. We say that $\alpha_1, \ldots, \alpha_n \in L$ form a transcendence basis for the extension if they algebraically independent over $K$ and $K(\alpha_1, \ldots, \alpha_n) \subseteq L$ is algebraic. The number $n$ is denoted by $\text{trdeg}_K(L)$.

By definition, for a $K$-algebra domain $A$, we set $\text{trdeg}_K(A) = \text{trdeg}_K(L)$, where $L$ is the fraction field of $A$.

Note that it is know that the notion of transcendence basis is well defined, and that any set of $K$-algebraically independent elements of $L$ can be completed to a transcendence basis of $L$ over $K$. Moreover, if $L = K(X)$ for some subset $X$ of $L$, then a transcendence basis of $L$ over $K$ can be extracted from $X$.

**Theorem 2.2.** Let $k$ be a field. Then $\dim(k[X_1, \ldots, X_n]) = n$.

**Proof.** Let $A = k[X_1, \ldots, X_n]$. Clearly

$$0 \subset (X_1) \subset \ldots \subset (X_1, \ldots, X_n)$$

is a strict chain of prime ideals so $\dim(A) \geq n$.

Claim: Let $P \subsetneq Q$ prime ideals in $A$. Then $\text{trdeg}_k(A/P) > \text{trdeg}_k(A/Q)$.

Since $A$ is a domain any maximal chain of prime ideals will have to start with the zero ideal. The above claim implies a maximal chain of length strictly greater than $n$ gives $\text{trdeg}_k(A) > n$. But it is well known that $k(X_1, \ldots, X_n)$ has transcendence degree equal to $n$. So all maximal chains of prime ideals in $A$ have length at most $n$, so $\dim(A) \leq n$.

Let us concentrate on proving the Claim.

Since $A/P$ maps onto $A/Q$ we see that $\text{trdeg}_k(A/P) \geq \text{trdeg}_k(A/Q) = r$. Assume that we have equality.

Clearly, if we denote the images of $X_i$ in $A/Q$ by $\alpha_i$, then $A/Q = k[\alpha_1, \ldots, \alpha_n]$. Because $k(\alpha_1, \ldots, \alpha_n)$ is the fraction field of $A/Q$, we can extract a transcendence basis for $A/Q$ over $k$ from $\{\alpha_1, \ldots, \alpha_n\}$, say $\alpha_1, \ldots, \alpha_r$. 
Let $u_i$ equal the image of $X_i$ in $A/P$, for $i = 1, \ldots, n$. Since $u_i$ maps to $\alpha_i$, we get that $u_1, \ldots, u_r$ are algebraically independent over $k$ as well so they form a transcendence basis of $A/P$ over $k$.

Let $S = k[X_1, \ldots, X_r] \setminus \{0\}$. Since $\alpha_1, \ldots, \alpha_r$ and $u_1, \ldots, u_r$ are algebraically independent over $k$ we get that $P \cap S = Q \cap S = \emptyset$.

Let us denote $K = k(X_1, \ldots, X_r)$. Then $S^{-1}A = K[X_{r+1}, \ldots, X_n]$. Moreover

$$\frac{S^{-1}A}{PS^{-1}A} \simeq k(u_1, \ldots, u_r)[u_{r+1}, \ldots, u_n].$$

But $u_{r+1}, \ldots, u_n$ are algebraic over $k(u_1, \ldots, u_r)$, so $k(u_1, \ldots, u_r)[u_{r+1}, \ldots, u_n]$ is in fact a field.

This gives that $PS^{-1}A$ is a maximal ideal in $S^{-1}A$, or, in other words, $P$ is a prime ideal of $A$ maximal with the property that $P \cap S = \emptyset$. This is clearly not true since $P \subsetneq Q$ and $Q \cap A = \emptyset$.

\[\square\]