LECTURE 10

1. Affine Space & the Zariski Topology

Definition 1.1. Let $k$ a field. Take $S$ a set of polynomials in $k[T_1, ..., T_n]$. Then $Z(S) = \{ x \in k^n \mid f(x) = 0, \forall f \in S \}$.

It is easy to check that $Z(S) = Z((S))$ with $(S)$ denoting the ideal generated by elements of $S$.

Definition 1.2. $Y \subseteq k^n$ is an (affine) algebraic set if $\exists S \subseteq k[T_1, ..., T_n] = A$ such that $Z(S) = Y$.

Example 1.3. (1) Consider the ideal $I = (xy) \subset k[x, y]$. Then $Z(I) = \{ (x, y) \in k^2 \mid xy = 0 \} = \{ x = 0 \} \cup \{ y = 0 \}$.

(2) Consider the ideal $I = (x^2 - y^3) \subset k[x, y]$. Then $Z(I) = \{ (x, y) \in k^2 \mid x^2 - y^3 = 0 \}$.

Proposition 1.4. The following are true:

1. The union of a finite collection of algebraic sets is algebraic.
2. Arbitrary intersections of algebraic sets are algebraic.
3. $\emptyset$ and $k^n$ are algebraic.

Proof. (1) It suffices to show this for the union of two sets. The general case can then be established by induction. Let $Y_1 = Z(T_1)$ and $Y_2 = Z(T_2)$. We claim that $Y_1 \cup Y_2 = Z((T_1)(T_2))$. For the forward inclusion, let $x \in Y_1 \cup Y_2$, so without loss of generality, assume $x \in Y_1$. This implies $f(x) = 0$ for all $f \in (T_1)$. Since $(T_1)(T_2) \subseteq (T_1)$, we have $f(x) = 0$ for every $f \in (T_1)(T_2)$ so $x \in Z((T_1)(T_2))$.

For the reverse inclusion, let $x \in Z((T_1)(T_2))$ so $h(x) = 0$ for all $h \in (T_1)(T_2)$. Assume $x \notin Z((T_2))$ so $\exists f \in (T_2)$ such that $f(x) \neq 0$. Take $g \in (T_1)$. Then $gf \in (T_1)(T_2)$ which implies $(gf)(x) = 0 \Rightarrow g(x)f(x) = 0$. But $f \neq 0$ so $g(x) = 0$ for every $g$. Thus $x \in Z((T_1)) = Y$, and we have equality of sets.

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The functions $Z(\lambda)$ defined on the family of ideals of the form $I(S)$ for some $S \subseteq \mathbb{A}^n$ and $I(\lambda)$ defined on algebraic sets in $\mathbb{A}^n$ are inverses to each other.

Theorem 1.10. (1) If $k = \mathbb{K}$, then $I(Z(J)) = \text{Rad}(J)$ for all $J \subseteq A = k[T_1, \ldots, T_n]$. This is known as the Hilbert Nullstellensatz.
(2) $Z(I(Y)) = \overline{Y}$ for all $Y \subseteq \mathbb{A}^n$.

Proof. The statement of (1) can be restated as $f \in I(Z(J) \iff f(P) = 0$ for all $P \in Z(J)$, where $Z(J) = \{x \in \mathbb{A}^n \mid g(x) = 0 \hspace{2mm} \forall \hspace{2mm} g \in J\}$. The statement implies that if $f$ vanishes where $J$ vanishes then $\exists h$ such that $f^h \in J$. This only follows when $k = \overline{k}$.

(2) We start with forward inclusion. Let $Y \subseteq Z(I(Y))$ which implies $\overline{Y} \subseteq Z(I(Y))$. For the reverse, let $W$ be a closed superset of $Y$. Then $W = Z(J)$, which gives $Z(J) \supseteq Y$. Examine the ideals corresponding to these sets, and we get $I(Y) \supseteq I(Z(J))$, so $J \subseteq I(Y)$. Now go to the sets corresponding to the ideals, and we get $Z(I(Y)) \subseteq Z(J) = W$, so any closed set containing $Y$ contains $Z(I(Y))$. This statement applied to $W = \overline{Y}$ gives that $\overline{Y} \supseteq Z(I(Y))$. □

Corollary 1.11. Assume that $k$ is an algebraically closed field. The maps $Z(\cdot)$ and $I(\cdot)$ are inverses to each other and establish a one-to-one correspondence between the family of algebraic sets in $\mathbb{A}^n$ and radical ideals of $A$.

Corollary 1.12. In this correspondence, a point $(a_1, \ldots, a_n) \in \mathbb{A}^n$ corresponds to the maximal ideal $(T_1 - a_1, \ldots, T_n - a_n)$ of $A$.

Proof. Let $I = (T_1 - a_1, \ldots, T_n - a_n)$ which is a maximal (hence radical) ideal.

The Corollary follows at once since $Z(I) = (a_1, \ldots, a_n)$. The correspondence implies that $I(a_1, \ldots, a_n) = (T_1 - a_1, \ldots, T_n - a_n)$ which is a non-trivial statement (and which may not be true if $k$ is not algebraically closed).

□

Definition 1.13. Let $\emptyset \neq Y \subseteq X$, with $X$ a topological space. Then $Y$ is irreducible if $Y$ is not a union of two proper closed subsets of $Y$.

An example of a reducible set in $\mathbb{A}^2$ is the set of points satisfying $xy = 0$ which is the union of the two axis of coordinates.

Definition 1.14. We call $Y$ an affine algebraic variety if $Y$ is an irreducible algebraic set.

Corollary 1.15. Let $Y$ be algebraic variety. Then $I(Y)$ is prime. Conversely, $I(Y)$ is prime implies that $Y$ is an algebraic variety. Therefore, in our 1-1 correspondence, varieties (irreducible algebraic sets) correspond to prime ideals.

Proof. Take $Y = Z(I)$ irreducible. Let $fg \in I(Y)$, so $(fg)(y) = 0$ for all $y \in Y$. This implies $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Then $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$. Note that both sets in our union
are closed in the subspace topology. But $Y$ is irreducible, so the sets $Y \cap Z(f)$ and $Y \cap Z(g)$ are not simultaneously proper. Assume, without loss of generality, that $Y \cap Z(f) = Y$ which implies that $Y \subseteq Z(f)$, so $f \in I(Y)$ by definition. This gives $f \in I(Y)$ and hence $I(Y)$ is prime.

Conversely, let $Y = Y_1 \cup Y_2$ with $Y_i = Z(I_i), I_i$ ideals in $A$, for $i = 1, 2$. Assume, for a contradiction, that $Y_1, Y_2$ are strictly contained in $Y$. First note that $I(Y) \subset I(Y_i)$ since equality would give $Z(I(Y_i)) = Z(I(Y))$, for $i = 1, 2$. But $Z(-), I(-)$ are inverses to each other when restricted to the set of ideals of algebraic sets and, respectively, algebraic sets, hence $Y_i = Y$, false.

Now, let $f_1 \in I(Y_1) \setminus I(Y)$. Then $f_1f_2 \in I(Y_1)I(Y_2) \subset I(Y_1) \cap I(Y_2) \subset I(Y_1 \cup Y_2) = I(Y)$.

But $I(Y)$ is prime. Therefore, either $f_1 \in I(Y)$ or $f_2 \in I(Y_2)$. This is a contradiction. 

\[\square\]

**Definition 1.16.** If $A = k[T_1, \ldots, T_n]$, and $Y \subseteq \mathbb{A}^n$ is an algebraic set, then $k[Y] = A/I(Y)$ is called the coordinate ring of functions of $Y$.

**Definition 1.17.** A map between two algebraic sets $\phi : V \subseteq \mathbb{A}^n \to W \subseteq \mathbb{A}^m$ is called a morphism (or regular map) if there are polynomials $F_1, \ldots, F_m$ such that

$$
\phi(a_1, \ldots, a_n) = (F_1(a_1, \ldots, a_n), \ldots, F_m(a_1, \ldots, a_n)),
$$

for all $(a_1, \ldots, a_n) \in V$.

A morphism $\phi$ is called isomorphism between $V$ and $W$ if there exists a morphism $\psi : W \to V$ inverse to $\phi$.

Let $\phi : V \subseteq \mathbb{A}^n \to W \subseteq \mathbb{A}^m$ be a morphism. This morphism induces a natural map $\phi_* : k[W] \to k[V]$ by $\phi_*(f) = f \circ \phi$.

Indeed, if $f - g \in I(W)$ then $f(w) = g(w)$ for all $w \in W$, so $f(\phi(v)) = g(\phi(v))$ for all $v \in V$, since $\phi(v) \in W$. This means that $f \circ \phi - g \circ \phi \in I(V)$ and hence $\phi_*$ is well defined. It is a routine check that $\Phi_*$ is in fact a $k$-algebra homomorphism.

Moreover, every $k$-algebra homomorphism $\Phi : k[W] \to k[V]$ is induced by a unique $\phi$, that is $\Phi = \phi_*$. The morphism $\phi$ is an isomorphism if and only if $\phi_*$ is an isomorphism of $k$-algebras.

Given $\Phi : k[W] \to k[V]$, $k$-algebra homomorphism, let us construct $\phi$:
Let $\Phi(\hat{T}_i) = \hat{F}_i$, for all $i = 1, \ldots, m$ and $F_i \in k[T_1, \ldots, T_n]$. Then $\phi = (F_1, \ldots, F_m)$ defines a morphism between $\mathbb{A}^n$ and $\mathbb{A}^m$. Let us show that it maps $V$ to $W$.

Let $g \in I(W)$ so $g(w) = 0$ for all $w \in W$. Moreover $g(\hat{T}_1, \ldots, \hat{T}_m) = g(T_1, \ldots, T_m) = \hat{0}$ in $k[W]$, since $g \in I(W)$.

Therefore $\Phi(g(\hat{T}_1, \ldots, \hat{T}_m)) = 0$ in $k[V]$ since a homomorphism maps 0 to 0.

But $\Phi$ is a $k$-algebra homomorphism, so $g(\Phi(\hat{T}_1), \ldots, \Phi(\hat{T}_m)) = 0$ which is equivalent to $g(\hat{F}_1, \ldots, \hat{F}_m) = 0$ in $k[V]$, or $g(F_1, \ldots, F_m) = 0$ in $k[V]$. This gives $g(F_1, \ldots, F_m) \in I(V)$ or in other words, $g(F_1, \ldots, F_m)(v) = 0$ for all $v \in V$, i.e. $g(F_1(v), \ldots, F_m(v)) = 0$ for all $v \in V$.

Since $\phi(v) = (F_1(v), \ldots F_m(v))$ we see that $g(\phi(v)) = 0$ for all $v \in V$ and so $g \in I(W)$ implies that $\phi(v) \in Z(g)$. In other words, $\phi(v) \in Z(I(W))$. But $Z(I(W)) = W$, since $W$ is an algebraic set, and so $\phi(V) \in W$.

Note that $\phi_*(\hat{T}_i) = \hat{T}_i \circ \phi = \hat{F}_i = \Phi(\hat{T}_i)$ for all $i = 1, \ldots, m$. Since $\hat{T}_i$ are $k$-algebra generators for $k[W]$ we get $\phi_* = \Phi$.

**Proposition 1.18.** Every nonempty affine algebraic set $V$ may be uniquely written in the form

$$V = V_1 \cup \cdots \cup V_n$$

where each $V_i$ is an algebraic variety and $V_i \not\subseteq V_j$ for all $j \neq i$. (These $V_i$'s are called irreducible components on $V$).

2. Dimension

**Definition 2.1.** Let $V \subseteq \mathbb{A}^n$ be an algebraic set. The supremum over all $n$ such that there exists a chain $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n$ of distinct irreducible algebraic sets in $V$ is called the dimension of $V$.

**Definition 2.2.** Let $P$ be a prime ideal in a ring $A$. The supremum over all $n$ such that there exists a chain of distinct prime ideals

$$P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n$$

contained in $P$ is called the height of $P$ and it is denoted by $ht(P)$.

The Krull dimension of $A$, $\dim(A)$, is the supremum of all $ht(P)$ over all $P \in \text{Spec}(A)$.

If $I$ is an arbitrary ideal of $A$, we let the height of $I$, $ht(I)$, equal the infimum of all $ht(P)$ over all prime ideals $P$ containing $I$. 
Theorem 2.3. Let $V$ be an algebraic set in $\mathbb{A}_k^n$ where $k$ is algebraically closed. Then

$$\dim(V) = \dim(k[V]).$$

Proof. This follows at once since irreducible algebraic sets in $V$ correspond to prime ideals containing $I(V)$ in our correspondence. \qed