Chapter 1: Lecture 11

1. The Spectrum of a Ring & the Zariski Topology

**Definition 1.1.** Let $A$ be a ring. For $I$ an ideal of $A$, define $V(I) = \{ P \in \text{Spec}(A) \mid I \subseteq P \}$.

**Proposition 1.2.** Let $A$ be a ring. Let $\Lambda$ be a set of indices and let $I_l$ denote ideals of $A$. Then

1. $V(0) = \text{Spec}(R), V(R) = \emptyset$;
2. $\bigcap_{l \in \Lambda} V(I_l) = V(\sum_{l \in \Lambda} I_l)$;
3. $\bigcup_{l=1}^{k} V(I_l) = V(\cap_{l=1}^{k} I_l)$;

Then the family of all sets of the form $V(I)$ with $I$ ideal in $A$ defines a topology on $\text{Spec}(A)$ where, by definition, each $V(I)$ is a closed set. We will call this topology the Zariski topology on $\text{Spec}(A)$.

Let $\text{Max}(A)$ be the set of maximal ideals in $A$. Since $\text{Max}(A) \subseteq \text{Spec}(A)$ we see that $\text{Max}(A)$ inherits the Zariski topology.

Now, let $k$ denote an algebraically closed field. Let $Y$ be an algebraic set in $A^n_k$. If we consider the Zariski topology on $Y \subseteq A^n_k$, then points of $Y$ correspond to maximal ideals in $A$ that contain $I(Y)$. That is, there is a natural homeomorphism between $Y$ and $\text{Max}(k[Y])$.

**Example 1.3.** Let $Y = \{(x, y) \mid x^2 = y^3\} \subset k^2$ with $k$ algebraically closed. Then points in $Y$ correspond to $\text{Max}\left(\frac{k[x, y]}{(x^2 - y^3)}\right) \subseteq \text{Spec}(\frac{k[x, y]}{(x^2 - y^3)})$. The latter set has a Zariski topology, and the restriction of the Zariski topology to the set of maximal ideals gives a topological space homeomorphic to $Y$ (with the Zariski topology).

Let $(A, \mathfrak{m}_A)$ be a local ring. Then $V(\mathfrak{m}_A)$ is the set of all prime ideals containing $\mathfrak{m}_A$, that is ideals equal to $\mathfrak{m}_A$. This implies that $\overline{\{\mathfrak{m}_A\}} = \{\mathfrak{m}_A\}$. Moreover, this shows that that a closed point $P$ in $\text{Spec}(A)$ (i.e., $\overline{\{P\}} = \{P\}$) is a maximal ideal of $A$.

Take $P_0 \leq A$, with $P_0$ a minimal prime ideal. Then $V(P_0) = \{ P \in \text{Spec}(A) \mid P_0 \subseteq P \}$ which is naturally identified with $\text{Spec}(A/P_0)$.

If $P_0 = 0$ with $A$ a domain, then $V(0) = \text{Spec}(A)$, and $\overline{\{0\}} = \text{Spec}(A)$. We call such a prime ideal the generic point of $\text{Spec}(A)$: $P \in \text{Spec}(A)$ such that $\text{Spec}(A) = \overline{\{P\}}$.

**Example 1.4.** Let $k = \overline{k}, A = k[x, y]$, and consider $\text{Spec}(A) \supseteq \text{Max}(A) = \{(x - \alpha, y - \beta) \mid \alpha, \beta \in k\}$. This latter set corresponds to $k^2$. 

1
\textbf{Definition 1.5.} Principal open sets are open sets of the form \(D(f) = \{ P \in \text{Spec}(A) \mid f \notin P \}\), for \(f \in A\). Note that this requirement is equivalent to \((f) \nsubseteq P\) or \(f \notin P\).

For \(f \in A\), \(D(f) \subseteq \text{Spec}(A)\). This implies that for \(P \in D(f)\) one has that \(P \cap \{1, f, f^2, \ldots, f^n, \ldots\} = \emptyset\), which implies that \(D(f)\) can be naturally identified with \(\text{Spec}(A_f)\).

If \(Y \subseteq \mathbb{A}^n\) is an algebraic set, and \(Y = Z(f)\) with \(f \in k[T_1, \ldots, T_n]\) then \(\mathbb{A}^n \setminus Y = \{ x \mid f(x) \neq 0 \}\). \(\text{Spec}(A_f)\) is the natural ring that corresponds to the principal open set \(D(f)\), since having \(f(x) \neq 0\) ‘implies’ that we can write \(f\) as a denominator.

Let \(f : A \to B\) be a ring homomorphism, and let \(p \in \text{Spec}(A)\), so \(f^{-1}(p) = p \cap A \in \text{Spec}(A)\). We can consider

\[\phi = f^* : \text{Spec}(B) \to \text{Spec}(A),\]

defined by \(\phi(p) = p \cap A\). If \(I \leq A\), then \(f^{-1}(V(I)) = V(IB)\). Note that \(\phi\) is therefore continuous, because a preimage under \(\phi\) of a closed set is a closed set.

However, if \(M \in \text{Max}(B)\), this does not imply that \(M \cap A = f^{-1}(M) = f^*(M) = \phi(M)\) belongs to \(\text{Max}(A)\).

Consider \(\pi\) the projection map from a surface onto a line. Take \(t\) a point on our line. Then \(\pi^{-1}(t)\) is a curve (not a point) on the surface so it is natural to study \(\text{Spec}(A)\), instead \(\text{Max}(A)\). (\(\text{Max}(A)\) would not allow us to pullback all the points, as \(\pi^{-1}(t)\) is not a point in general.) The curve does define a prime ideal.

\textbf{Definition 1.6.} Let \(\phi\) be defined as above, and take \(p \in \text{Spec}(A)\). The \textit{fiber over} \(p\) is defined to be \(\phi^{-1}(p) = \{ Q \in \text{Spec}(B) \mid Q \cap A = p \}\).

It can be shown that \(\phi^{-1}(p)\) is homeomorphic to \(\text{Spec}(B \otimes_A \frac{A_p}{pA_p})\). The ring \(\frac{A_p}{pA_p}\) is a field since it is a quotient of a local ring by its maximal ideal.

\textbf{Definition 1.7.} If \(p \in \text{Spec}(A)\), then we call \(\frac{A_p}{pA_p}\) the \textit{residue field at} \(p\), denoted \(k(p)\). One can see that this field is also the \textit{fraction field of} \(A/p\).

We can now see the following string of equalities:

\[B \otimes_A k_p = B \otimes_A \frac{A_p}{pA_p} = \frac{B \otimes_A A_p}{p(B \otimes_A A_p)} = \frac{B_p}{pB_p} = \frac{B}{pB}(A\setminus p)\).

Let us compute \(\text{Spec}(B \otimes_A k(p))\). We know that \(S = A \setminus P \subseteq A \subseteq B \to B/pB\), so \(Q/pB \cap S = \emptyset\). The first term in our intersection is in \(\text{Spec}(B/pB)\), and hence \(\text{Spec}(B \otimes_A k(p)) = \).
\{ Q \in \text{Spec}(B) \mid Q \cap (A \setminus P) = \emptyset, Q \supseteq pB \} = \{ p \in \text{Spec}(B) \mid P \cap A = p, P \supseteq pB \} = \{ p \in \text{Spec}(B) \mid P \cap A = p \} = \phi^{-1}(p)$. So it is natural to call $B \otimes_A k(p)$ the fiber ring at $p$.

1.1. Exercises.

(1) Let $M$ be an $A$-module, and let $x \in M$. If $x = 0$ in $M_m$ for every $m \in \text{Max}(A)$, then $x = 0$.

Proof. Assume $x \neq 0$, which implies $\text{Ann}_A(x) \not\subseteq A$. Then $\text{Ann}_A(x) \subseteq m$ for some $m \in \text{Max}(A)$. But $x = 0$ in $M_m$ so $\exists u \in A \setminus m$ such that $ux = 0$, which implies $u \in \text{Ann}_A(x) \subseteq m$, a contradiction. Hence $x = 0$. Note that $M_m$ is a module over the local ring $A_m$. □

(2) Let $M$ be a finitely generated $A$-module. If $M \otimes_A k(m) = 0$ for all $m \in \text{Max}(A)$ then $M = 0$.

Proof. (Sketch) We know $M \otimes_A k(m) \cong \frac{M_m}{mm_m}$, so $M_m = mM_m$ for all $m \in \text{Max}(A)$. Now apply NAK and exercise 1. □

2. Krull Dimension

**Definition 2.1.** Let $P \in \text{Spec}(A)$. A chain of prime ideals descending from $P$ of length $m$ is:

$$P_0 \subset P_1 \subset \cdots \subset P_m = P$$

with each $P_i \in \text{Spec}(A)$.

**Definition 2.2.** We define the height of $P$, denoted $ht(P)$ to be the supremum of the lengths of chains of prime ideals descending from $P$.

**Definition 2.3.** The Krull dimension of a ring $A$, denoted $\dim(A)$, is defined to be the supremum of $ht(P)$, where $P$ runs over all prime ideals in $\text{Spec}(A)$.

We can have $\dim(A) = \infty$, even if $A$ is Noetherian.

2.1. Some examples. If $k$ is a field, then $\dim(k) = 0$.

If $A$ is a PID, then $\dim(A) = 1$. Some examples of PID’s and chains of prime ideals of length one are: $0 \subseteq (x) \subseteq k[x]$, and $0 \subseteq (p) \subseteq \mathbb{Z}$.

To a prime ideal $P$, we associate an algebraic set that is defined by taking common the zeroes of all polynomials in $P$. Each ideal in the chain descending from $P$ correspond to subsets of the original algebraic set. We now have the following chains,
point $\subseteq$ some curve $\subseteq$ some plane $\subseteq \cdots \subseteq$ whole variety

Maximal ideal $\supseteq \cdots \supseteq (0),$

where single points are the zero sets for maximal ideals, and the whole space is the zero set for the zero polynomial.

**Remark 2.4.** (1) If $Y$ is algebraic variety in $\mathbb{A}^n$, one can define the *codimension* of $Y$ as the supremum of all $n$ such that $Y = Y_0 \subset Y_1 \subset \cdots \subset Y_n$ of strict chains of irreducible algebraic varieties in $\mathbb{A}^n$. This is denoted by $\text{codim}(Y)$.

If $Y$ is an algebraic set, then $\text{codim}(Y)$ is by definition the infimum of all $\text{codim}(Z)$ where $Z$ is an irreducible component of $Y$.

(2) Let $x \in Y$, where $Y$ is an algebraic set in $\mathbb{A}^n$. The dimension of $Y$ at $x$ is by definition the infimum of $\dim(U)$ where $U$ runs over all open sets in $Y$ containing $x$.

### 3. Integral Extensions

**Definition 3.1.** If we have a map $R \hookrightarrow S$, then $s \in S$ is integral over $R$ if there exist $n$ and $r_0, \ldots, r_n \in R$ such that

$$x^n + r_n s^{n-1} + \cdots + r_1 s + r_0 = 0.$$  

Some Facts:

1. An algebra over a field $k$ is module finite over $k$ if and only if it is a finite dimensional vector space over $k$. Also, $R \subseteq S$ is module finite if $\exists s_1, \ldots, s_n \in S$ such that $S = Rs_1 + \cdots + Rs_n$.

2. $K \subseteq L$ a field extension is module finite if and only if it is a finite algebraic extension.

**Example 3.2.** $\mathbb{Z}\left[\frac{1}{2}\right] = \{ \frac{n}{2^k} \mid n \in \mathbb{Z}, \ k \in \mathbb{W} \}$ is not module finite over $\mathbb{Z}$ since $\mathbb{Z}\left[\frac{1}{2}\right] \neq \mathbb{Z}\frac{1}{2} + \mathbb{Z}\frac{1}{2^2} + \cdots + \mathbb{Z}\frac{1}{2^n}$ because we cannot get higher powers of 2 in the denominator.

**Example 3.3.** $\mathbb{Z}\left[\sqrt{2}\right]$ is module finite over $\mathbb{Z}$.

In general, $R/I$ is module finite over $R$ for all $I \subseteq R$: the projection map can be written as $\pi : R \rightarrow R/I = R\overline{T} = R(\pi(1))$ when we consider $\pi(a) = \overline{a} = a + I$ and so $R/I$ si $R$-spanned by $\pi(1)$.

**Remark 3.4.** One can talk of $\phi : R \rightarrow S$ as an integral map by regarding $S$ as an $R$-algebra via $\phi$, that is by reducing to an extension $R/\ker(\phi) = \text{Im}(\phi) \hookrightarrow S$.

We will show:
1. $R \hookrightarrow S$ integral implies $\dim R = \dim S$.

2. Every finitely generated $k$-algebra is module finite over a polynomial ring over $k$. This is known as the Noether Normalization Theorem.

**Theorem 3.5.** If $R \hookrightarrow S$, then the following are equivalent:

1. $S$ is a finitely generated $R$-algebra and every $s \in S$ is integral over $R$.
2. $S = R[v_1,...,v_n]$ where $v_i$ are integral over $R$.
3. $S$ is module finite over $R$.

**Proof.** (1) implies (2) is trivial.

For (2) implies (3), we proceed by induction on $n$. If $n = 1$, then $S = R[v_1]$ with $v_1$ integral over $R$ so $\exists m$ such that $v_1^m + r_{m-1}v_1^{m-1} + \cdots + r_1v + r_0 = 0$ with $r_i \in R$. This implies that $v^m \in (1, v, ..., v^{n-1})_R$. We also have that $v^{m+1} + r_{m-1}v^m + \cdots + r_1v^2 + r_0v = 0$ which implies $v^{n+1} \in (1, v, ..., v^{n-1})_R$ for all $k \geq m$, so $S = R + Rv + \cdots + Rv^{m-1}$. For $n > 1$, we have $T = R[v_1, ..., v_{n-1}]$ is module finite over $R$. Let $S = T[v_n]$. We can say that $R \subseteq T$ is module finite with $t_1, ..., t_k$ as generators of $R$ over $T$. The fact that $T \subseteq S$ is also module finite implies that $R \subseteq S$ is module finite by transitivity: If $s_1, ..., s_n$ generate $S$ over $T$, then $s_it_j$ generate $S$ over $R$.

For (3) implies (1), let $S = Rs_1 + \cdots + Rs_n$. Clearly $S = R[s_1,...,s_n]$. Need $s \in S$ integral over $R$. If $s_1 \neq 1$, then put 1 in the list of generators and rename $s_1 = 1$. For all $i$, we have $ss_i = \sum_{j=1}^{n} v_{ij}s_j$, with $v_{ij} \in R$. Then $A = sI_n - (r_{ij}) \in M_n(S)$, so the product of $A$ with the column vector consisting of all the $s$’s must be 0. Multiply by $\text{Adj}(A)$, to obtain that $\det(A)$ times this column vector equals 0. Since $s_1 = 1$, we have $\det(A)s_1 = \det(A) = 0$. The form of $A = sI_n - (r_{ij})$ shows that $\det(A)$ is a monic polynomial expression in $s$ with coefficients in $R$, and thus $s$ is integral over $R$.

**Proposition 3.6.** 1. If $R \subseteq S$, then the integral elements of $S$ form a subring called the integral closure of $R$ in $S$.

**Proof.** Let $t, s \in S$ with $t$ and $s$ integral in $S$. Then $R[s, t]$ is integral over $R$ by previous theorem, and $s + t, st \in R[s, t]$ so $s + t$ and $st$ are integral.

**Definition 3.7.** We call $R \hookrightarrow S$ an integral extension if every $s \in S$ is integral over $R$.

**Definition 3.8.** Let $R$ be a domain, and $R \subseteq L$ where $L$ is a field. The integral closure of $R$ in $L$ is denoted by $R_L$ (or $\overline{R_L}$). If $L = Q(R)$, the field of fractions of $R$, then $R'_Q(R)$, called the integral closure of $R$, is denoted by $R'$ or $\overline{R}$. 