LECTURE 19

1. Properties of completion; Artin-Rees lemma

It is helpful to note that any element of \( \hat{A}^I \) is given by a sequence \( \{x_n\} \) such that 
\[ x_{n+1} - x_n \in I^n. \]
Hence we can find \( a_{n+1} \in I^n \) for all \( n \geq 0 \) such that 
\[ x_n = a_1 + \cdots + a_n \]
for all \( n \geq 1 \).

It can easily be checked that if \( A = R[x_1, \ldots, X_n] \) is a polynomial ring over a ring \( R \),
and \( I = (X_1, \ldots, X_n) \), then \( \hat{A}^I = R[[X_1, \ldots, X_n]] \).

**Proposition 1.1.** Let \( A, B \) be two rings and let \( I \) be an ideal of \( A \), respectively \( J \) be an ideal of \( B \). Consider \( f : A \to B \) be a ring homomorphism such that \( f(I) \subset J \). Then there is a canonical ring homomorphism \( \hat{f} : \hat{A}^I \to \hat{B}^J \).

Moreover, if \( f \) is surjective such that \( f(I) = J \), then \( \hat{f} \) is surjective.

**Proof.** We have natural maps \( A/I^n \to B/J^n \), so \( \lim \rightarrow A/I^n \to B/J^n \) for all \( n \) which
implies, by applying the universal property of the inverse limit, the first part.

For the second part, consider a sequence of elements in \( B \), say \( \{b_n\} \) such that \( b_{n+1} \in J^n \),
\[ y_n = b_1 + \cdots + b_n \] and let \( \{y_n\}_n \) give an elements in \( \hat{B}^J \). But \( I^n \) maps onto \( J^n \) via \( f \), so
we can find a sequence of elements \( a_{n+1} \in I^n \) mappint onto \( b_{n+1} \). Set \( x_n = a_1 + \cdots + a_n \),
for \( n \geq 1 \). Then \( \{x_n\}_n \) gives an element in \( \hat{A}^I \) that maps onto the element corresponding
to \( \{y_n\}_n \) in \( \hat{B}^J \). \( \square \)

**Corollary 1.2.** If \( A \) is Noetherian and \( I = (r_1, \ldots, r_n) \subset A \), then \( \hat{A}^I \) is Noetherian.

**Remark 1.3.** In fact, we hane \( \hat{A}^I = \frac{A[[X_1, \ldots, X_n]]}{(X_i - r_i, \ldots, X_n - r_n)} \). This fact will be proved later.

**Proof.** Indeed, let \( I = (r_1, \ldots, r_n) \). Map \( R[[X_1, \ldots, X_n]] \) onto \( A \) by sending \( X_i \to r_i \).
This maps \( (X_1, \ldots, X_n) \) onto \( I \) and hence we obtain \( \hat{A}^I \) as a quotient of the Noetherian
ring \( A[[X_1, \ldots, X_n]] \). \( \square \)
Theorem 1.4. Let $A$ be a ring and $I$ and ideal of $A$. Let $\pi : \hat{A}^I \to A/I$ the natural projection. Then $\text{Ker}(\pi) \subseteq \text{Jac}(\hat{A}^I)$. This implies that there is a one-to-one correspondence between the maximal ideals in $\hat{A}^I$ and the maximal ideal of $A/I$. In particular, the completion of a local ring $(A, \mathfrak{m})$ at its maximal ideal is a local ring as well.

Proof. Let $\{x_n\}$ an element $x$ of $\hat{A}^I$ that belongs to $\text{Ker}(\pi)$: $x_n \in I$ for all $n$. We will show that $1 + x$ is invertible in $\hat{A}^I$. Consider $y_n = \sum_{i=0}^{n+1} (-1)^i x_n^i$. It is clear that $y_n$ define a Cauchy sequence in $A$ which therefore gives an element $y$ of the completion. But then $z_n = 1 - (1 + x_n)y_n = x_n^{n+2} \in I^{n+2}$. This implies that $\{z_n\}_n$ is 0 in $\hat{A}^I$ and then $1 = (1 + x)y$ in $\hat{A}^I$. □

Definition 1.5. Let $A$ be a ring, $I$ an ideal of $A$ and $M$ an $A$-module. We say that a sequence of elements $\{x_n\}_n$ in $M$ is Cauchy in the $I$-adic topology if for all $n$ there exists $N$ such that $x_i - x_j \in I^n M$ for all $i, j \geq N$. A sequence $\{x_n\}_n$ of elements from $M$ converges to 0 if for all $n$ there exists $N$ such that $x_i \in I^n M$ for all $i \geq N$. A sequence $\{x_n\}_n$ converges to an element $x \in M$ such that $\{x_n - x\}_n$ converges to zero in $M$. We say that $M$ is complete in the $I$-adic topology if every Cauchy sequence in $M$ converges to an element in $M$. We say that $M$ is $I$-adically separated if $\cap_{n=1}^{\infty} I^n M = 0$.

Definition 1.6. The $I$-adic completion of $M$ is $\hat{M}^I : \varprojlim M/I^n M$. It can be checked that $\hat{M}^I$ is a $\hat{A}^I$-module and there exists a natural $A$-module homomorphism $M \to \hat{M}^I$ with kernel $\cap_n I^n M$.

We say that a filtration of submodules of $M$ say $\{N_n\}$ is cofinal with the filtration $\{I^n M\}$ if for all $n$ there exists $m$ such that $N_m \subseteq I^n M$ and for all $t$ there exists $s$ such that $I^s M \subseteq N_t$. It can be checked that $\varprojlim M/N_n \simeq \varprojlim M/I^n M$ (in fact, the filtrations define the same linear topology on $M$).

Moreover, we can see that a Cauchy sequence and a subsequence of it define the same element in $\hat{M}^I$, so we assume that every element $m \in \hat{M}$ is defined by a sequence $\{m_n\}$ such that $m_{n+1} - m_n \in I^n$. Therefore there exists $z_{n+1} \in I^n$ such that for $y_n = z_0 + \ldots + z_n$ we have that $\{y_n\}_n$ gives $m$.

Proposition 1.7. Let $A$ be a ring, $I$ an ideal of $A$. Then

1. Any $A$-linear map $f : M \to N$ of $A$-modules induces an $\hat{A}^I$-linear map $\hat{f} : \hat{M}^I \to \hat{N}^I$. Moreover, $f$ surjective implies that $\hat{f}$ is surjective.
(2) There exists a natural isomorphism of \( \hat{A}^I \)-modules \( \hat{M}^I \oplus \hat{N}^I \simeq \hat{M}^I \oplus \hat{N}^I \), for any two \( A \)-modules \( M, N \).

(3) The multiplication by an element \( M \overset{a}{\to} N \) defines a natural map \( \hat{A}^I \)-linear map \( \hat{M}^I \to \hat{N}^I \) given by the multiplication by the image of \( a \in \hat{A}^I \).

Proof. The proof of the first part follows the ring case mutatis mutandis. The last two parts are straightforward \( \square \)

Let \( N \subset M \) be a pair of \( A \)-modules. In what follows we need to compare the \( I \)-adic topology on \( N \) with the topology induced by the \( I \)-adic topology on \( M \) restricted to \( N \). In essence we will show that \( \lim \underset{\leftarrow}{\text{N}}/I^n M \cap N = \hat{N}^I \). To prove this we need to develop some considerations on filtrations of modules and in fact we will be proving a statement that is more general.

**Definition 1.8.** Let \( M \) be an \( A \)-module and \( I \) an ideal of \( A \). Let \( \mathcal{M} = \{M_n\}_n \) be a filtration of submodules of \( M \), i.e. \( M_{n+1} \subset M_n \) and \( M_0 = M \). We say that \( \mathcal{M} \) is an \( I \)-filtration if \( IM_n \subset M_{n+1} \) for all \( n \geq 0 \). The filtration \( \mathcal{M} \) is called \( I \)-stable if \( I^n M_m = M_{n+1} \) for \( n \gg 0 \).

An example of an \( I \)-stable filtration is the one given by \( \{I^n M\}_n \). The case of \( M = A \) is particularly important because we can associate the following object to the filtration \( \{I^n\}_n \): \( gr_I(A) := I^n/I^{n+1} \) which is an \( A \)-module naturally. In fact this object, which is called the *associated graded ring* with respect to the ideal \( I \) is a ring with multiplication defined as follows: \( \overline{ab} = \overline{a} \overline{b} \) for any two elements \( a \in I^n, b \in I^m \). It can be checked that this is well-defined and that it extends via distributivity to a multiplication on \( gr_I(A) \).

Now consider an \( I \)-filtration \( \mathcal{M} \). We can define the following \( A \)-module \( gr_{\mathcal{M}}(M) : = \bigoplus_{n \geq 0} M_n/M_{n+1} \). An important feature of it is that this object is in fact an \( gr_I(A) \)-module. For \( \overline{a} \in I^n/I^{n+1} \) and \( \overline{m} \in M_k/M_{k+1} \), we let \( \overline{am} : = \overline{a} \overline{m} \in M_{n+k}/M_{n+k+1} \). It can be checked that definition is well-defined. By distributivity, we can extend this to a scalar multiplication on \( gr_{\mathcal{M}}(M) \) with elements from \( gr_I(A) \) and we call it the *associated graded module* of \( M \) with respect to \( \mathcal{M} \).

**Proposition 1.9.** Let \( A \) be a ring, \( I \) be an ideal of \( A \), \( M \) be an \( A \)-module, and \( \mathcal{M} \) be an \( I \)-filtration on \( M \). Then
(1) If $A[It]$ is a finitely generated $A$-algebra, if $I$ is a finitely generated ideal.

(2) If $A[It]/IA[It] \simeq gr_I(A)$ as $A$-algebras.

Proof. For (1), let $I = (a_1, \ldots, a_r)$. Then $A[X_1, \ldots, x_n]$ maps onto $A[It]$ under $X_i \mapsto a_i$.

For (2), let $A[It] \to gr_I(A)$ that sends $a \in I/I^2$, for any $a \in I$. It can be easily check that this is an well-defined $A$-algebra homomorphism with kernel equal to $IA[It]$.

\[\square\]

Proposition 1.10. Let $A$ be a ring, $I$ be an ideal of $A$, $M$ be a finitely generated module over $A$, and $\mathcal{M}$ be an $I$-stable filtration on $M$ composed of finitely generated submodules.

Then $gr_{\mathcal{M}}(M)$ is a finitely generated module over $gr_I(A)$.

Proof. Since the filtration is $I$-stable so $I^k M_N = M_{N+k}$ for some $N \geq 0$ and all $k \geq 0$. Therefore $\frac{M_n}{M_{n+1}} = \frac{M_{n+1}}{M_{n+2}}$, for all $n \geq N$.

This shows that $gr_{\mathcal{M}}(M)$ is generated by the union of all the generators of $M_n/M_{n+1}$ for $n \leq N$. This is a finite set which proves the claim. \[\square\]

Definition 1.11. Let $I$ be an ideal in $A$. The $A$-algebra $R_I(A) = A[It] \subset A[t]$ is called the Rees algebra, or the blowup algebra, of $A$ with respect to $I$.

Note that $A[It] = \oplus_{n \geq 0} I^n$.

Similarly, for an $I$-filtration $\mathcal{M}$ on an $A$-module $M$, we can define the Rees module of $M$ with respect to $\mathcal{M}$ by $R_{\mathcal{M}}(M) := \oplus_{n \geq 0} M_n t^n = \oplus_{n \geq 0} M_n$. Note that $R_{\mathcal{M}}(M)$ is a module over $A[It]$ in a natural way.

Theorem 1.12. Let $A$ be a ring, $I$ be an ideal of $A$, $M$ be an $A$-module with $I$-filtration $\mathcal{M}$ consisting of finitely generated $A$-submodules of $M$. Then the filtration $\mathcal{M}$ is $I$-stable if and only if $R_{\mathcal{M}}(M)$ is a finitely generated $A[It]$-module.

Proof. If $\mathcal{M}$ is $I$-stable, then $M_{N+k} = I^k M_N$ for some $N \geq 0$ and for all $k \geq 0$. Then $R_{\mathcal{M}}(M)$ is finitely generated by the union of the generators of $M_i$, with $I \leq N$.

If $R_{\mathcal{M}}(M)$ is finitely generated over $A[It]$, there exists $N \geq 0$ such that all generators belong to the union of $M_i$, $i \leq N$. But $\oplus_{k \geq 0} M_{N+k}$ is finitely generated as an $A[It]$-module (since it is a homomorphic image of $R_{\mathcal{M}}(M)$). Using this and the fact that $\mathcal{M}$ is an $I$-filtration we derive that $I^k M_N = M_{N+k}$, for all $k \geq 0$. \[\square\]
Corollary 1.13 (Artin-Rees Lemma). Let $M$ be a finitely generated $A$-module, where $A$ is Noetherian. Assume that $I$ is an ideal of $A$ and let $N$ an $A$-submodule of $M$.

1. Let $\mathcal{M}$ be an $I$-stable filtration on $M$. Then $\{M_n \cap N\}_n$ is an $I$-stable filtration on $N$.

2. The filtration $\{I^n M \cap N\}_n$ is $I$-stable that is there exists $c > 0$ such that

$$I^n M \cap N = I^{n-c}(I^c M \cap N),$$

for all $n \geq c$.

Proof. It suffices to prove (1). Let $\mathcal{M}'$ the filtration with terms $I^n M \cap N$. Clearly $R_{\mathcal{M}'}(N)$ is an $A[It]$-submodule of $R_{\mathcal{M}}(M)$.

Note that $A[It]$ is a Noetherian $A$-algebra and $R_{\mathcal{M}}(M)$ is a finitely generated $A[It]$-module by Theorem 1.12. So, $R_{\mathcal{M}'}(N)$ is a finitely generated $A[It]$-module, hence by Theorem 1.12 we get that $\mathcal{M}'$ is $I$-stable on $N$. 

Proof. To be included later.

Theorem 1.14. Let $A$ be a Noetherian ring and $I$ an ideal of $A$.

1. If $0 \to N \to M \to P \to 0$ is a short exact sequence of finitely generated $A$-modules then

$$0 \to \hat{N}^I \to \hat{M}^I \to \hat{P}^I \to 0$$

is a short exact sequence of $\hat{A}^I$-modules.

2. The universal property of the tensor product implies that there is a natural $\hat{A}^I$-map $\hat{A}^I \otimes_A M \to \hat{M}^I$ for any $A$-module $M$. For every finitely generated $A$-module $M$, $\hat{A}^I \otimes_A M \simeq \hat{M}^I$ is an isomorphism.

3. $\hat{A}^I$ is a flat $A$-algebras which is faithfully flat if $(A, \mathfrak{m})$ is a local ring.

Proof. To be included later.