Chapter 1: Lecture 5

If $A$ is a ring and $M$ an $A$-module we let

$$\mathcal{L}_A(M) = \{ N \subseteq M : N \text{ is an } A - \text{submodule of } M \}.$$ 

Let $N$ be an $A$-submodule of $M$. We refer to it as proper if $N \neq M$. If $N = 0$ or $M$, then we say that $N$ is trivial.

**Theorem 0.1** (Krull). If $A$ is finitely generated and $L$ is a proper submodule of $M$, then there is a maximal proper submodule of $M$, say $N$, with $L \leq N$.

**Proof.** Let $P = \mathcal{L}_A(M) \setminus \{ M \}$ and consider $(P, \subseteq)$ which is a partially ordered set. We need to show that every nonempty totally ordered subset of $P$ has an upper bound. We intend to use Zorn’s Lemma so it suffices to show this is true for chains.

Let $N_1 \leq N_2 \leq \cdots$ be a chain in $P$. Then $N = \bigcup_{i=1}^{\infty} N_i$ is an $A$-submodule of $M$ which is the desired upper bound. It is proper: if $N = M$, then, since $M$ is finitely generated, say by $x_1, \ldots, x_r$, and $P$ is a chain, then there exists $N_i$ such that $x_1, \ldots, x_r$ are all contained in $N_i$. But this forces $N_i = M$ which is not the case.

So, Zorn’s Lemma can be applied and the statement of the theorem follows.

□

**Definition 0.2.** Let $M$ be an $A$-module. We call $M$ simple if it does not contain nontrivial $A$-submodules.

It is clear that if $M$ is simple then, for each $x \neq 0$ in $M$, the $A$-submodule $Ax$ must equal $M$. Also, a simple $A$-module $M$ satisfies both the DCC and ACC conditions in trivial fashion. Let $M$ be an arbitrary $A$-module. Let $N, N' \in \mathcal{L}_A(M)$, with $N \subseteq N'$. By the Correspondence Theorem for quotients, $N'/N$ is $A$-simple if and only if there are no submodules $K \in \mathcal{L}_A(M)$ such that $N \subsetneq K \subsetneq N'$.

With this observation, we remark that under the notations and hypotheses of the above Theorem, the module $M/N$ is $A$-simple.

Also, if $M$ happens to contain a minimal nonzero submodule $N$, then $N$ is $A$-simple. We can always pick such a submodule if $M$ is Artinian over $A$. 

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Finally, let $S$ A-simple, and $x \neq 0$ an element of $A$. Since $S = Ax$, then the map $A \to Ax$ that sends $a$ to $ax$ gives that $A/\text{Ann}(x) \simeq Ax = S$. Therefore, $S$ simple is equivalent to the condition that $\text{Ann}(x)$ is a maximal ideal of $A$.

**Definition 0.3.** Let $M$ be an $A$-module. An ascending filtration of $M$ is a family of submodules $\{M_i\}$ such that $M_i \subseteq M_{i+1}$. The $A$-modules $M_{i+1}/M_i$ are called the factors of the filtration. In similar fashion, one can define descending filtrations. Filtrations can be finite or infinite. Obviously, any ascending filtration can be renumbered to become a descending filtration, and conversely.

**Definition 0.4.** Consider $\_A M$. Then $M$ is said to be of **finite length** over $A$ if there is a chain of $A$-submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that $\frac{M_i}{M_{i-1}}$ is $A$-simple for every $i$.

Such a chain will be called a composition series of length $n$ for $M$, and $\frac{M_i}{M_{i-1}}, i = 1, \ldots, n$, are its factors.

**Proposition 0.5.** Let $A$ be a ring and $\_A M$ a module. Then $\_A M$ is of finite length if and only if $M$ is Artinian and Noetherian.

**Proof.** First assume that $M$ is of finite length and let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a chain of submodules such that $\frac{M_i}{M_{i-1}}$ are $A$-simple for every $i$, i.e., each one has no proper submodules.

If $S$ is a simple module over $A$, then $S$ is Noetherian and Artinian. This remark applies to all the factors of the composition series.

Consider

$$0 \to M_i \to M_{i+1} \to M_{i+1}/M_i \to 0.$$  

Using the behavior of Noetherian/Artinian modules on short exact sequences, we can show by induction on $i$ that every $M_i$ is Noetherian and so $M = M_n$ is Noetherian.

Conversely, let us take $M$ Artinian and Noetherian and $M_0 = 0$. Let $M_1$ minimal among the nonzero submodules of $M$. If $M_1 \neq M$, let $M_2$ minimal among the submodules of $M$ strictly containing $M_1$ and so on. For all these steps we use that $M$ is Artinian. We have an ascending chain of submodules $\{M_i\}$ and clearly $M_{i+1}/M_i$ are all simple submodules. $M$ is Noetherian and hence the chain must stabilize. By construction it must stabilize at $M_n = M$, for some $n$, and this shows that $M$ is of finite length. $\square$
We plan to prove that all composition series of a finite length module have the same length. We will do so by proving a stronger statement on composition series. For this, we first need a lemma.

**Lemma 0.6.** Let \( \Lambda M \) a module and \( E, F, G \in \mathcal{L}_A(M) \) such that \( E \subsetneq F \) and \( F/E \) is \( A \)-simple. Then \( \frac{F \cap G}{E \cap G} \) is either zero or \( A \)-simple.

**Proof.** The natural projection \( F \to F/E \) induces, by restriction, an \( A \)-linear map \( F \cap G \to F/E \). The kernel of this map equals \( E \cap G \).

Hence \( \frac{F \cap G}{E \cap G} \) injects into \( \frac{F}{E} \). Therefore the image is either zero or the entire \( \frac{F}{E} \) since this is an simple \( A \)-module. \( \square \)

**Theorem 0.7** (Jordan-Hölder). Let \( \Lambda M \) be a module with two finite composition series:

\[
0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M
\]

and

\[
0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = M.
\]

Then \( m = n \) and there exists a permutation \( \sigma \in S_n \) such that \( M_{i+1}/M_i \simeq N_{\sigma(i)+1}/N_{\sigma(i)} \) for all \( i = 0, \ldots, n - 1 \).

**Proof.** If \( M \) is simple then the statement is immediately true. Let us assume that the statement is true for all \( A \)-modules that admit at least one composition series of length \( n - 1 \).

Consider the filtration:

\[
0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_n \cap N_{m-1} = N_{m-1}.
\]

Its factors are either zero or simple by Lemma 0.6. So the induction hypothesis applies to \( N_{m-1} \).

Note that we also have the following filtration

\[
0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_{n-1}.
\]

Its factors are either zero or simple by Lemma 0.6 except possibly the last factor. The induction hypothesis applies to \( M_{m-1} \) as well.
Case A: If $M_{n-1} = N_{m-1}$ then it follows that $n - 1 = m - 1$ and $M_{i+1}/M_i \simeq N_{\sigma(i)+1}/N_{\sigma(i)}$ for all $i = 0, \ldots, n - 2$ and some $\sigma \in S_{n-1}$. Note that $M/M_{n-1} = M/N_{n-1}$. Putting this together gets us the statement.

Case B: If $M_{n-1} \neq N_{m-1}$, since $M_{n-1}, N_{m-1}$ are proper maximal submodules of $M$ we see that $M_{n-1} + N_{m-1} = M$.

Note that by using the third isomorphism theorem for modules

\[
\frac{M}{N_{m-1}} = \frac{M_{n-1} + N_{m-1}}{N_{m-1}} \simeq \frac{M_{n-1}}{M_{n-1} \cap N_{m-1}}.
\]

Therefore $\frac{M_{n-1}}{M_{n-1} \cap N_{m-1}}$ is $A$-simple as well. This guarantees that

\[0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_{n-1}\]

is a composition series of $M_{n-1}$ after removing the redundant terms corresponding to possible zero factors.

Now note that by the induction hypothesis $M_{n-1}$ satisfies the statement of the theorem, and hence all its composition series must have length $n - 1$. So, in fact, exactly one term is redundant in the above filtration. After removing this factor we obtain as a consequence that

\[0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_{n-1} \subseteq M\]

is a composition series of length $n - 2$ and completing it at the end with $N_{m-1}$ we obtain a composition series for $N_{m-1}$ of length $n - 1$. But the induction hypothesis applied to $N_{m-1}$ shows that all its composition series must have length $m - 1$, so $n - 1 = m - 1$ or $n = m$.

For the remaining part the reader should keep in mind that $n = m$.

Let us also note that

\[
\frac{M}{M_{n-1}} = \frac{M_{n-1} + N_{m-1}}{M_{n-1}} \simeq \frac{N_{m-1}}{M_{n-1} \cap N_{m-1}}.
\]

This (and the similar isomorphism proved earlier) allows us to conclude that the two resulting composition series

\[0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_{n-1} \subseteq M_n = M\]

and
\[ 0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq N_{m-1} \subseteq N_m = M \]

have the same factors up to a permutation.

But

\[ 0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq M_{n-1} \subseteq M_n = M \]

has the same factors, up to a permutation, as

\[ 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M \]

by Case A which was already considered.

Similarly

\[ 0 = M_0 \cap N_{m-1} \subseteq M_1 \cap N_{m-1} \subseteq \cdots \subseteq M_{n-1} \cap N_{m-1} \subseteq N_{m-1} \subset N_m = M \]

has the same factors, up to a permutation, as

\[ 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{m-1} \subseteq N_m = M. \]

Putting all these together we get our statement.


Example 0.8. Let \( A = \frac{k[x]}{(x^3)} \). The chain of ideals \( 0 \subset (x^2) \subset (x) \subset A \) is a composition series for \( A \) as an \( A \)-module.

Definition 0.9. For an \( A \)-module \( M \) of finite length, the length of any composition series is call the length of \( M \). It is denoted by \( \lambda_A(M) \) or, simply, \( \lambda(M) \).

An important result is that the length function behave nicely on short exact sequences of finite length modules.

Proposition 0.10. Let \( 0 \to N \to M \to K \to 0 \) be a short exact sequence of \( A \)-modules of finite length. Then

\[ \lambda(M) = \lambda(N) + \lambda(K). \]

Proof. We can consider that \( N \) is a submodule of \( M \) and note that \( K \simeq M/N \). Hence any composition series of \( K \) can be lifted back to \( M \) resulting in a filtration of \( N \subseteq M \) with \( A \)-simple
factors. We can concatenate it to a composition series for $N$ and obtain a composition series for $M$. Counting the numbers of factors gives the statement of the Proposition.

\[\square\]

A few corollaries can be obtained readily.

**Corollary 0.11.** Let $M_i$ be $A$-modules, $i = 1, \ldots, n$, of finite length.

Then $M = \oplus_{i=1}^n M_i$ is of finite length over $A$ and $\lambda(M) = \sum_{i=1}^n \lambda(M_i)$.

**Proof.** Consider the following short exact sequences for $k = 2, \ldots, n$:

\[
0 \to \oplus_{i=1}^{k-1} kM_i \to \oplus_{i=1}^k M_i \to M_k.
\]

Apply Propositions 0.5 and 0.10 to get the result. \[\square\]

**Theorem 0.12.** Let $A$ be an Artinian ring and $M$ and finitely generated $A$-module. Then $M$ has finite length over $A$.

**Proof.** We will prove this by induction on the number of generators of $M$.

If $M$ is cyclic then $M = Ax$, $x \in M$ and then $M \simeq A/I$, where $I = \text{Ann}(x)$. Since $A$ is Artinian and Noetherian it follows that $A$ is of finite length over $A$. Hence $A/I$ is of finite length over $A$ which proves that $M$ is of finite length as well.

Now, let $x_1, \ldots, x_{n-1}, x_n$ be generators for $M$, where $n > 1$. Then let $N = Rx_1 + \cdots + Rx_{n-1}$ and consider

\[
0 \to N \to M \to M/N \to 0.
\]

Remark that the image of $x_n$ in $M/N$ generates $M/N$. Since $N, M/N$ are both of finite length by the induction hypothesis, we can conclude that $M$ is of finite length.

\[\square\]