Applications of pseudocanonical covers to tight closure problems

Florian Enescu *

Abstract. It is well known that nice conditions on the canonical module of a ring have a strong impact in the study of strong F-regularity and F-purity. For example, in the Gorenstein (or even $\mathbb{Q}$-Gorenstein) case strong F-regularity is equivalent to weak F-regularity, and it is conjectured that this is true in general ([10], [13]). In this note, we will show how to use the double cover of a ring to obtain sufficient conditions for strong F-regularity and F-purity. Our results involve a closure operation for a pair of ideals that could be of relevance for the conjecture mentioned above.

0 Introduction

The theory of cyclic covers has proven to be a powerfool tool in algebraic geometry and it has been often used in the study of various classes of singularities (see [6], [12], [14], [21]). Recently, some problems arising in tight closure theory have been approached by looking at canonical or anticanonical covers (both are special types of cyclic covers) as in the papers by I. Aberbach, M. Katzman, B. MacCrimmon, N. Hara and K. E. Smith, A. K. Singh, K.-i. Watanabe ([1], [2], [8], [13], [15], [17], [18], [22], [24]). Some authors believe there is an intimate connection between singularities in birational geometry and tight closure theory ([7], [17] and [19]). With this in mind, using cyclic covers in tight closure problems should appear as no surprise.

The purpose of this note is to show how a construction that is a slight variation of a cyclic cover (called here *pseudocanonical double cover*) can be used in studying concepts as strong F-regularity and F-purity. The novelty stays in this being the first time when a cyclic cover other that the canonical or the anticanonical ones is used in tight closure problems. Moreover, we do not require the canonical module to be a torsion element in the divisor class group as

*Address: University of Utah, Department of Mathematics, Salt Lake City, Utah 84112, USA and Institute of Mathematics of the Romanian Academy. e-mail: enescu@math.utah.edu.

for the canonical cover or impose any Noetherian hypothesis on auxiliary rings as in the anticanonical cover case. Although our results give only sufficient conditions for strong F-regularity and F-purity, they have the advantage that they are easy to use and do not refer back to the cover just introduced. In addition to this, they generalize and put in a new perspective the well known fact that strong F-regularity and F-purity deform for Gorenstein rings.

In Section 1 we will introduce the notion of a pseudocanonical double cover and list its properties that are of interest to us. The reader should be aware that the theory of normal cyclic covers has beed studied extensively in the paper by M. Tomari and K.-i. Watanabe ([20]). For a detailed account of the theory the reader should refer to that paper where the language of divisors is used. Since our results are sligthly more general and we do not use their point of view in our note, we do not assume that readers are familiar with cyclic covers.

In Section 2, the main results are stated. It is proven that whenever a Cohen-Macaulay normal local ring modulo a canonical ideal gives an F-injective or F-rational (in the F-finite case) ring, then it is F-pure or strongly F-regular, respectively (for complete statements, see Corollaries 2.5 and 2.9. Some examples and concluding remarks are also given.

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1 The pseudocanonical double cover

Let (R, m, K) be a local, Cohen-Macaulay domain of prime characteristic p that admits a canonical module ω_R. Denote by I an ideal in R that is isomorphic to the canonical module ω_R. Since the beginning of tight closure theory, the canonical module has played an important role through local duality or via constructions that involve it. When the local ring R is $\mathbb{Q}$-Gorenstein (that is, when the canonical module is a torsion element in the divisor class group), tight closure theory for the ring is better understood. For a $\mathbb{Q}$-Gorenstein ring R one can define its *canonical cover* which is a split extension that carries important information on R with it. By definition, the canonical cover of a normal ring R is the ring

$$S_{can} = R \oplus I \oplus I^{(2)} \oplus ... \oplus I^{(n-1)}$$

where the identification $I^{(n)} \simeq uR$ gives the natural ring structure on S, and n is the order of I in the divisor class group of R. It is known that if n is not divisible by p, then R is strongly F-regular if and only if S_{can} is (see for example [23] or [18]).

The anticanonical cover of a normal ring R is defined similarly. Let J be an ideal of pure height 1 in R that is the inverse of the canonical module of R in the divisor class group. The anticanonical cover of R is the ring

$$S_{acan} = \oplus_{i \geq 0} J^{(i)} t^i.$$

This ring need not be Noetherian in general. A theorem of K.-i. Watanabe says that if S_{acan} is Noetherian, then R is strongly F-regular (F-pure) if and only if S_{acan} is strongly F-regular (F-pure) ([24]).

Here, we will use a different construction that resembles the usual canonical covers to study certain problems in tight closure theory.

Let (R, m, K) be a local ring as above. One can embed the canonical module as an ideal $I \subset R$. If R is Gorenstein then I can be taken to be any principal ideal, and therefore one can take it to be the whole ring. If R is not Gorenstein, then I is a height one ideal such that R/I is Gorenstein. Fix $f \in m$, an arbitray element, and let t be an indeterminate. On the R-module $S = R \oplus_R It$, one obtains an R-algebra structure by defining $t^2 := f$. Thus, S can be thought of as the subring of $R[T]/(T^2 - f)$ consisting of elements of the form $R + It$, where t is the image of some indeterminate T. Since $R[T]/(T^2 - f)$ is R-free on the basis $1, t$, $S \simeq R \oplus_R I$.

Definition 1.1. We will call the ring S defined above *a pseudocanonical double cover* of R. This construction depends on the choice of $f \in m$, although the properties that interest us will be independent of this choice.

Let us list some of the properties of pseudocanonical double covers.

Proposition 1.2. *Let S be a pseudocanonical cover for R. Then:*

i) S is a local ring with maximal ideal equal to $m \oplus It$. S is a domain if f is square-free in the fraction field of R;

ii) If $\underline{x} = x_1, ..., x_n$ form a regular sequence on R, they form a regular sequence on S, too. Moreover, S is a Gorenstein ring. The socle generator for $S/\underline{x}S$ is given by ut where $u \in I$ is such that its image gives the socle generator of $I/\underline{x}I$.

Proof. i) Let $a + bt$ be an element not in $m \oplus It$. That is, $a \notin m$. To show that there are $c, d \in R$ such that $(a + bt)(c + dt) = 1$ we need to solve the following system over R

$$ac + bdf = 1,$$

$$ad + bc = 0$$

where the unknowns are c and d. Regarding the system over the fraction field of R, one gets that

$$c = \frac{a}{b^2 f - a^2},$$

$$d = \frac{b}{b^2 f - a^2}.$$

It can be easily checked that $b^2 f - a^2$ is a unit in R, hence both c and d belong in fact to R.

For the last statement of i), by looking at the equation $(a + bt)(c + dt) = 0$, we obtain a similar system of equations

$$ac + bdf = 0,$$

$$ad + bc = 0$$

with unknowns c and d. Solving that over the fraction field of R one gets that both c and d are zero (under the assumption that either a or b are nonzero), unless $b^2 f - a^2 = 0$. But this contradicts that f is square-free in the fraction field of R.

ii) First, it is easy to see that if $x \in R$, then $S/xS \simeq R/xR \oplus I/xI$. To prove now the part regarding regular sequences, we proceed by induction. The case $n = 1$ follows as in the proof of i) and the induction step is trivial, since if I is a canonical ideal for R, then I/xI is a canonical ideal for R/xR.

To show that S is Gorenstein first note that S is a finite extension of R and therefore $\omega_S \simeq \mathrm{Hom}_R(S, \omega_R)$. By local duality, $\mathrm{Hom}_R(S, \omega_R) \simeq R \oplus \omega_R$ as R-modules and it can be easily seen that this isomorphism preserves the S-action. So, $\omega_S \simeq S$. For the last part, note that $0 \neq u$ belongs to $\mathrm{Soc}(I/\underline{x}I)$ and hence $0 \neq ut$ belongs to $\mathrm{Soc}(S)$.

□

In the next section we will show how this construction can be used to formulate sufficient conditions for certain maps that are at the core of tight closure theory to split.

2 Main Result

First, we would like to remind the reader a few definitions from tight closure theory.

Definition 2.1. Let R be a reduced Noetherian ring.

1. We say that R is *strongly F-regular* if it is F-finite and for every c, not in any minimal prime of R, there exist q such that the R-linear map $\varphi_c : R \longrightarrow R^{1/q}$ sending 1 to $c^{1/q}$ splits as a map of R-modules.

2. If there exists q such that the R-linear map $\varphi_1 : R \longrightarrow R^{1/q}$ sending 1 to 1 splits as a map of R-modules, we say R is *F-split*. An F-split ring is F-pure and the converse holds for F-finite rings.

The following well-known result of M. Hochster ([9]) will be very useful in proving our main theorem.

Theorem 2.2. *(Hochster) Let (R, m, K) be a Gorenstein local ring and M a finitely generated R-module. Assume that $x_1, ..., x_d$ form a system of parameters on R, $u \in R$ is the socle generator of $R/(x_1, ..., x_d)$, and let $m \in M$ a nonzerodivisor on M. Then the R-linear map φ $R \to M$ that sends 1 to m splits over R if and only if, for every integer $k \geq 0$, $(x_1 \cdots x_d)^k um \notin (x_1^{k+1}, ..., x_d^{k+1})M$. Moreover, if* $\mathrm{depth}(M) = d$ *then φ splits over R if and only if $um \notin (x_1, ..., x_d)M$.*

Proof. For a proof see Remark 2 in [9]. □

Now we can state the main result of this note:

Theorem 2.3. *Let (R, m, K) a local Cohen-Macaulay domain of characteristic $p > 0$ and let $c \in R$ be a nonzero element. Assume that R admits a canonical module ω_R and fix I an ideal isomorphic to ω_R. Denote by J the ideal generated by $\underline{x} = x_1, ..., x_n$ that form a regular sequence in R, where $n = \dim(R)$. Let $u \in R$ such that its image generates the socle in I/JI. Assume that $cu^q \in J^{[q]}I$ for every $q \gg 0$ implies that $u \in JI$. Then the map $\varphi_c : R \to R^{1/q}$ that sends 1 to $c^{1/q}$ splits for some q.*

Proof. Using the same notations as in the preceding section, let us consider S a pseudocanonical double cover of R.

Fix $q \gg 0$ and consider the following map $\psi_c : S = R + It \longrightarrow R^{1/q} + I^{1/q}t$ sending 1 to $c^{1/q}$ and extending it to an S-linear map.

We want to show that ψ_c splits over S. We have seen earlier that S is Gorenstein and $\underline{x}$ form a system of parameters in S. The socle generator of S/JS is ut.

The map ψ_c splits if $c^{1/q}ut \notin J(R^{1/q} + I^{1/q}t)$, because S is Gorenstein and $\underline{x}$ form a regular sequence in S (see Theorem 2.2).

On the other hand if $c^{1/q}ut \in J(R^{1/q} + I^{1/q}t)$ we get that

$$cu^q \in J^{[q]}I.$$

If this happens for every $q \gg 0$, then u belongs to JI, according to the hypothesis. This is impossible since u has been chosen not in JI to begin with (see Proposition 1.2 ii)).

So, there exists some $q \gg 0$ for which the map ψ_c splits. Let us say this splitting is given by $T : R^{1/q} + I^{1/q}t \to S = R + It$. Then by restricting T to $R^{1/q}$ and composing it to the canonical projection $\pi : R + It \to R$ we get a splitting for φ_c. Indeed, the composition is R-linear and $\pi T(\varphi_c(1)) = \pi T(c^{1/q}) = \pi(1) = 1$. This concludes the proof.

□

The following proposition shows how the result can be applied indicating an interesting case when the hypotheses of Theorem 2.3 are fulfilled (for $c = 1$).

Proposition 2.4. *Let (R, m, K) be a local Cohen-Macaulay domain with canonical module ω_R. Denote by I an ideal of R that is isomorphic to ω_R. Suppose $\underline{x} = x_1, ..., x_n$ form a system of parameters in R, and denote the ideal that they generate by J. Assume that*

i) R/I is F-injective;

ii) $x_1 \in I$ and $x_2, ..., x_n$ form a regular sequence on R/I.

If $u \in J \cap I$ such that $u^q \in J^{[q]}I$ for every $q \gg 0$, then $u \in JI$.

Proof. Since $u \in J$ one can write $u = \sum_{i=1}^{n} a_i x_i = a_1 x_1 + \sum_{i \geq 2} a_i x_i$. According to the hypothesis we have that

$$a_1^q x_1^q + \sum_{i \geq 2} a_i^q x_i^q = \sum_{i=1}^{n} b_i x_i^q,$$

where each b_i belongs to I.

Hence, $(\overline{a_1}^q - \overline{b_1})\overline{x_1}^q = 0$ in $R/(x_2^q, ..., x_n^q)$. Since R is Cohen-Macaulay, the above relation implies that $a_1^q - b_1 \in (x_2^q, ..., x_n^q)$.

So, $a_1^q \in I + (x_2^q, ..., x_n^q)$ for every $q >> 0$. Since R/I is F -injective we get that $a_1 \in I + (x_2, ..., x_n)$. In conclusion, $a_1x_1 \in Ix_1 + (x_2, ..., x_n)x_1 \subset JI$.

Now, $u - a_1x_1 = \sum_{i \geq 2} a_i x_i \in (x_2, ..., x_n) \cap I = (x_2, ..., x_n)I$. The last equality follows since $x_2, ..., x_n$ form a regular sequence on R/I. So, $u - a_1x_1 \in JI$ and therefore $u \in JI$.

□

Corollary 2.5. *Let (R, m, K) be a local Cohen-Macaulay domain with canonical module ω_R. Assume that one can embed ω_R in R as a proper ideal $I \subset R$ such that R/I is F-injective. Then R is F-split.*

Proof. We can assume that R is not Gorenstein. Since I is isomorphic to the canonical module of R we have that its height equals 1. Let us choose $x_1 \in I$ a nonzero divisor. R/I is Gorenstein, so we can choose $x_2, ..., x_n$ such that they form a regular sequence on R/I and $x_1, ..., x_n$ form a regular sequence on R. Denote by J the ideal generated by $(x_1, .., x_n)$ and take $u \in I$ such that its image generates the socle of I/JI. We would like to show that $u \in J \cap I$, so we need to show that $u \in J$. But $mu \subset JI$, so in fact u kills the module I/JI. This is a faithful R/J-module, therefore $u \in J$. Assume that $u^q \in J^{[q]}I$, for every $q \gg 0$. According to the previous Proposition we have that $u \in JI$. Now, we can apply Theorem 2.3 to get that R is F-split. □

We can apply the same ideas to obtain sufficient conditions for the local ring R to be strongly F-regular. First, we need the following Proposition which coupled with Theorem 2.3 will allow us to state the result referring to strong F-regularity.

Proposition 2.6. *Replace condition i) by i') R/I is F-rational and keep the rest of notations and assumptions of Proposition 2.4 unchanged. If $u \in J \cap I$ such that there exists $c \notin I$ with $cu^q \in J^{[q]}I$ for every $q \gg 0$, then $u \in JI$.*

Proof. The proof is similar to the one for Proposition 2.4. □

In similar manner as above, we can state the following

Corollary 2.7. *Let (R, m, K) be an F-finite local Cohen-Macaulay domain with canonical module ω_R. Assume that one can embed ω_R in R as a proper ideal $I \subset R$ such that R/I is F-rational. Then, for every $c \notin I$, the map $\varphi_c : R \to R^{1/q}$ sending 1 to $c^{1/q}$ splits for some $q \gg 0$.*

Remark 2.8. Although the definition of strong F-regularity requires to have a splitting for every c not in any minimal prime of R, it is enough to check this condition for only one element c such that R_c is strongly F-regular (due to a result by Hochster and Huneke ([10]).

Corollary 2.9. *Let (R, m, K) be an F-finite local Cohen-Macaulay domain with canonical module ω_R. Assume that one can embed ω_R in R as a proper ideal $I \subset R$ such that R/I is F-rational. Assume that $I \in \mathrm{Reg}(R)$ = the set of primes where R_P is regular (this happens for example when R is normal). Then R is strongly F-regular.*

Proof. Cover $\mathrm{Spec}(R)$ with principal affine open sets of the form $D(f)$ where $f \in R$. Then there exists $f \in R$ such that $I \in D(f) \subset \mathrm{Reg}(R)$. That means that $f \notin I$ and R_f is regular, hence strongly F-regular, and we can apply Corollary 2.7 having in mind the preceding remark. □

Remark 2.10. 1. It should be noted that in both Corollaries 2.5 and 2.9 the ring R/I is Gorenstein and has dimension one less than the dimension of R, therefore its tight closure properties should be easier to understand than those of the original ring R.

2. It is well known that for a Gorenstein ring R, if x is a nonzerodivisor on R such that R/xR is strongly F-regular (respectively F-pure), then R is strongly F-regular (F-pure). In fact, the assertion regarding strong F-regularity is known to extend to $\mathbb{Q}$-Gorenstein rings ([1], [13];see also [16] for the general case). On the other hand, in a Gorenstein ring, a principal ideal is isomorphic to the canonical module, therefore one can see our Corollaries 2.5 and 2.9 as another way of generalizing the Gorenstein case.

3. Ian M. Aberbach has shown that if (R, m) is F-finite, weakly F-regular and R/I is F-rational, then R is strongly F-regular (see Theorem 2.2.2 and the discussion preceding it in [3]). Our Corollary 2.9 provides a natural generalization of this result.

We will ilustrate now our results with a few examples.

Example 2.11. 1. Let $R = k[ax, bx, ay, by, az, bz] \subset k[a, b, x, y, z]$, where k is a perfect field of characteristic $p > 0$. This ring is local, Cohen-Macaulay and not Gorenstein. Its canonical module equals $(ax, bx)R$. It can be checked easily that $R/(ax, bx)R$ is F-rational. To see this, one can apply Fedder's criterion to show that this quotient ring is F-pure. Then, the F-rationality follows from noting that this ring has negative a-invariant. Our Corollary 2.6 then implies that R is strongly F-regular.

2. A more difficult example is the following. For $k \geq 4$, $t \geq 4$ consider $X_1, X_2, X_3, X_4, Y_1, ..., Y_k, Z_1, ..., Z_t$ indeterminates. Let K be a perfect field of characteristic $p > 0$ and take $F_1, ..., F_4$ and $G_1, ..., G_4$ part of a system of parameters in $K[Y's]$, and $K[Z's]$ respectively. Assume that $F_1F_4 - F_2F_3$ and $G_1G_4 - G_2G_3$ both define F-rational singularities in $K[Y's]$ and $K[Z's]$, respectively.

Consider the matrix

$$\begin{pmatrix} F_1 & F_2 & 0 & 0 \\ F_3 & F_4 & 0 & 0 \\ X_1 & X_2 & G_1 & G_2 \\ X_3 & X_4 & G_3 & G_4 \end{pmatrix}$$

Denote by I_2 the ideal generated in $K[X's, Y's, Z's]$ by the 2-minors that involve only nonzero entries of the matrix. Let

$$R = K[X's, Y's, Z's]/I_2.$$

We claim that this ring is strongly F-regular.

Let us compute first a canonical ideal for R. Denote by $\underline{p}$ the ideal generated in R by the images of $X_1, \ldots, X_4$. We claim that $\underline{p}$ is a canonical ideal for R. To see this introduce a new ring

$$T = K[X's, F's, G's]/I_2.$$

In this ring we can regard the $F's$ and the $G's$ as new variables and therefore T is a one-sided ladder determinantal ring as in [4]. Using Conca's result that describes the canonical module of a one-sided ladder determinantal ring (Theorem 4.2 in [4]), we obtain that the images of $X_1, \ldots, X_4$ in T generate a canonical ideal. But R is a flat extension of T with Gorenstein closed fiber and this shows that the images of $X_1, \ldots, X_4$ in R give the canonical ideal in R.

To prove that R is strongly F-regular it remains to be shown that $R/\underline{p}$ is an F-rational ring. But $R/\underline{p}$ is just the tensor product of two F-rational rings over a perfect field and this is an F-rational ring (see for example [5]).

To get a specific example, take $k = t = 4$, $p \neq 3$, $F_1 = Y_1$, $F_2 = Y_2$, $F_3 = Y_2^2 + Y_3$, $F_4 = Y_1^2 + Y_4$, $G_1 = Z_1$, $G_2 = Z_2$, $G_3 = Z_2^2 + Z_3$, $G_4 = Z_1^2 + Z_4$. Now, $F_1F_4 - F_2F_3$ and $G_1G_4 - G_2G_3$ both define F-rational singularities in $K[Y's]$ and $K[Z's]$, respectively, because they have isolated singularities and negative a-invariant.

We end the paper with the following concluding

Remark 2.12. One can define a mixed tight closure operation for a pair of two ideals I and J in R in the following way. We will say that an element $u \in J \cap I$ is in the mixed tight closure of J and I if there is c, not in any minimal prime of R, such that $cu^q \in J^{[q]}I$ for every $q \gg 0$. We denote this closure by $(J, I)^*$. It is clear from our Theorem 2.3 that a good understanding of such a closure operation will have implications in the study of strong F-regularity. We have in mind the important conjecture in tight closure theory which states that strong F-regularity is equivalent to weak F-regularity for F-finite rings. One of the difficulties that one faces is that this closure operation is not very nice unless more conditions are imposed on the ideals J and I. For our purposes, it suffices that J be generated by a system of parameters and I be isomorphic to ω_R. In the view of Theorem 2.3, for a weakly F-regular ring R we would like to choose I radical ideal such that $(J, I)^* = JI$. This would imply that R is strongly F-regular. This appears to be a difficult problem even when we assume that the ring we start with is strongly F-regular.

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