F-injective rings and F-stable primes*

Florian Enescu †

Abstract. The notion of stability of the highest local cohomology module with respect to the Frobenius functor originates in the work of R. Hartshorne and R. Speiser. R. Fedder and K.-i. Watanabe examined this concept for isolated singularities by relating it to F-rationality. The purpose of this note is to study what happens in the case of non-isolated singularities and to show how this stability concept encapsulates a few of the subtleties of tight closure theory. Our study can be seen as a generalization of the work by Fedder and Watanabe. We introduce two new ring invariants, the F-stability number and the set of F-stable primes. We associate to every ideal I generated by a system of parameters and $x \in I^* - I$ an ideal of multipliers denoted $I(x)$ and obtain a family of ideals $Z_{I,R}$. The set $\mathrm{Max}(Z_{I,R})$ is independent of I and consists of finitely many prime ideals. It also equals $\mathrm{Max}\{P | P$ prime ideal such that R_P is F-stable$\}$. The maximal height of such primes defines the F-stability number.

1 The F-stability number

We will start by introducing the notations and terminology that will be used in the paper.

All the rings will be of positive characteristic $p > 0$ and everywhere in these notes q denotes a power of the characteristic. By a quasi-local ring (R, m, K) we mean a ring with only one maximal ideal m, such that $R/m = K$. For a Noetherian ring R, R^0 denotes the complement of the union of all the minimal primes of R. A local ring is a Noetherian quasi-local ring. For an ideal I, the ideal generated by the q^{th} powers of its elements will be denoted by $I^{[q]}$. An element $x \in R$ belongs to the tight closure of I, I^*, if there is $c \in R^0$ such that $cx^q \in I^{[q]}$, for all sufficiently large $q = p^e$. Similarly, $x \in R$ belongs to the Frobenius closure of I, I^F, if $x^q \in I^{[q]}$, for all sufficiently

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†University of Utah, Department of Mathematics, 1400 East, 155 South, Salt Lake City, UT 84112 and Institute of Mathematics of the Romanian Academy, Bucharest, Romania; e-mail address: enescu@math.utah.edu.

large $q = p^e$. By a parameter test element for R we mean an element $c \in R^0$ such that for every parameter ideal I and every $x \in I^*$, $cx^q \in I^{[q]}$, for all $q = p^e$. The ideal generated by the set of parameter test elements will be called the parameter test ideal and be denoted by $\tau_p(R)$.

A Noetherian ring of characteristic p is called *F-rational* if every parameter ideal I is tightly closed, that is $I^* = I$. A Noetherian ring R is *F-pure* if the homomorphism $R \longrightarrow R^{(1)}$ is pure, i.e. $M \longrightarrow R^{(1)} \otimes_R M$ is injective for every R -module M (here, $R^{(1)}$ is regarded as an R-algebra via the Frobenius homomorphism). This implies that every ideal I is Frobenius closed, i.e. $I = I^F$. A Noetherian ring R is *F-injective* if the Frobenius map induces an injective map of local cohomology of modules $H^i_m(R) \longrightarrow H^i_m(R^{(1)})$, for every $0 \leq i \leq \dim(R)$. For Cohen-Macaulay rings this is equivalent to all the parameter ideals being Frobenius closed.

Throughout this note (R, m, K) will be a local, reduced, Cohen-Macaulay ring of characteristic p and Krull dimension d.

Definition 1.1. Consider $x_1, ..., x_d$ a system of parameters for R, and write $I = (x_1, ..., x_d)$ for the ideal generated by them. For every $x \notin I$, we can define the ideal $I(x) := \{c \in R | cx^q \in I^{[q]}$, where $q = p^e$, for all $e \gg 0\}$.

Remark 1.2. In tight closure terms, the ideal $I(x)$ consists of the elements c that work in a tight closure test for x and I. The definition of $I(x)$ also makes sense for $x \in I$. However, this case is not interesting and we exclude it for expository reasons.

A similar concept has been studied by K. E. Smith in [7], where she has considered the ideals generated by test elements that work for one fixed ideal I, *independent* of the choice of $x \in I^*$. More precisely, she has defined the *test ideal for I* as $\tau(I) = \cap_{q \gg 0}(I^{[q]} :_R (I^*)^{[q]})$ where the intersection over $q \gg 0$ is interpreted as follows. For any family of ideals $\{I_q\}_{q=p^e>0}$, one can define another family of ideals $\{\cap_{q \geq t} I_q\}$ that gives an increasing chain of ideals. Since R is Noetherian, the chain stabilizes and this stable ideal is denoted by $\cap_{q \gg 0} I_q$ (see [7]). It can be checked that, whenever $I \neq I^*$, $\tau(I) = \cap_{x \in I^* \setminus I} I(x)$.

It is easy to see that $I(x) = R$ if and only if $x \in I^F \setminus I$. Also for $x \notin I$, $x \in I^*$ if and only if $I(x) \cap R^o \neq \varnothing$. Therefore, $x \notin I^*$ if and only if there is $\underline{p} \in \operatorname{Min}(R)$ such that $I(x) \subseteq \underline{p}$ if and only if $\operatorname{ht}(I(x)) = 0$. This says that the ideals $I(x)$ are interesting only if x belongs to I^* and not to I^F.

Now we can define the F-stability number of R with respect to I.

Definition 1.3. Define the *F-stability number of R with respect to I* as $\max\{\operatorname{ht}(I(x)) | x \notin I\}$ and denote this number by $\operatorname{fs}_I(R)$.

The definition suggests that this number depends upon the choice of the ideal I. In fact, the following lemma shows that $\mathrm{fs}_I(R)$ is independent of the choice of I.

Lemma 1.4. *(see [2]) Let (R, m) be as above (i.e., local, reduced and Cohen-Macaulay). Also, let $(x_1, ..., x_d)$ be a system of parameters for (R, m) and let $f \in R$ such that η is the image of $\overline{f} \in R/(x_1, ..., x_d)$ in $H^d_m(R)$. Then, for fixed $N > 0$ and $Q = p^N$,*

$$\cap_{e>N}(0 :_R F^e(\eta)) = \cap_{q>Q}(I^{[q]} : f^q).$$

Here, $I = (x_1, ..., x_d)$.

Proposition 1.5. *The F-stability number is independent of the choice of the system of parameters.*

Proof. Let I and J be two ideals generated by full systems of parameters and write $H^d_m(R) = \varinjlim R/I^{[q]} = \varinjlim R/J^{[q]}$.

Choose $x \notin I$ such that $\mathrm{fs}(R) = \mathrm{ht}I(x)$. According to the previous Lemma, $I(x) = \cap_{e\gg 0}(0 :_R F^e(\eta))$ where η is the image in $H^d_m(R)$ of the class of x in R/I. In fact, a nonzero multiple of η, say γ, belongs to the socle of $H^d_m(R)$. Clearly, $\cap_{e\gg 0}(0 :_R F^e(\eta)) \subset \cap_{e\gg 0}(0 :_R F^e(\gamma))$. Moreover, one can find an element $y \in R/J$ that maps to γ in $H^d_m(R)$. This is true because, for every s.o.p. $y_1, ..., y_d$ that generates J, the injective map $R/(y_1^t, ..., y_d^t) \to R/(y_1^{t+1}, ..., y_d^{t+1})$ induces an isomorphism between the corresponding socles (as it has been remarked in [2], page 235, the point is that these socles are R/m-vector spaces of the same dimension).

In conclusion, applying Lemma 1.4 again, we get that $I(x) \subset J(y)$ and $y \notin J$, so $\mathrm{fs}_I(R) \leq \mathrm{fs}_J(R)$.

By reversing the roles of I and J, we have proved that $\mathrm{fs}_I(R) = \mathrm{fs}_J(R)$, and the statement of the proposition follows.

□

From now on we denote the F-stability number of R by $\mathrm{fs}(R)$.

Remark 1.6. It is easy to see that R is F-rational if and only if $\mathrm{fs}(R) = 0$. R is F-injective if and only if $\mathrm{fs}(R)$ is finite.

In what follows we will relate this new invariant to the work of R. Fedder and K.-i. Watanabe ([2]).

Theorem 1.7. *(Hochster-Huneke) Let $f \in R^o$ be such that R_f is regular. Assume that R is F-finite or essentially of finite type over an excellent local ring. Then there exists $d \geq 0$ such that for any ideal I and any $x \in I^*$, $f^d \in I(x)$.*

Corollary 1.8. *Let (R, m, K) be a local ring that is F-finite or essentially of finite type over an excellent local ring. Assume also that R has an isolated singularity. Then* $\mathrm{fs}(R) \in \{0, \dim(R), \infty\}$.

Proof. Suppose $\mathrm{fs}(R) \neq 0$, and $\mathrm{fs}(R) \neq \infty$. So, R is not F-rational and R is F-injective. Choose $x \in I^* \setminus I$ such that $\mathrm{fs}(R) = \mathrm{ht}(I(x))$. According to the above theorem, for every $f \in m \cap R^o$ there is some d such that $f^d \in I(x)$. Therefore, f belongs to all minimal primes over $I(x)$. Since R is equidimensional, it follows that $\mathrm{ht}(m) = \mathrm{ht}(I(x))$ and we are done. □

Proposition 1.9. *Let $x \in R$ be a nonzerodivisor. Then* $\mathrm{fs}(R) \leq \mathrm{fs}(R/xR) + 1$, *unless* $\mathrm{fs}(R/xR) = 0$, *when it follows that* $\mathrm{fs}(R) = 0$.

Proof. Extend x to a system of parameters in R, say $x, x_2, ..., x_d$ and choose $z \in R$, not in I, such that $\mathrm{ht}(I(z)) = \mathrm{fs}(R)$. For every $c \in I(z)$ we have that $cz^q \in (x, x_2, ..., x_d)^{[q]}R$, for $q \gg 0$. Assume that $c \neq 0$ and write $c = dx^t$, where d is not in xR. It is easy to see that $dz^q \in (x_2, ..., x_d)^{[q]}R$, mod xR (here we use that R is Cohen-Macaulay and so $x, x_2, ..., x_d$ form a regular sequence).

If $\overline{z} \in (\overline{x_2}, ..., \overline{x_d})$ in R/xR, then $z \in I$ which is impossible. So, $\overline{d} \in I(\overline{z})$ in R/xR. Denote by J the preimage of $I(\overline{z})$ back in R. Hence, $d \in J \setminus xR$, and $c \in J$. This shows that $\mathrm{fs}(R) = \mathrm{ht}(I(z)) \leq \mathrm{ht}(J) = \mathrm{fs}(R/xR) + 1$ and establishes the inequality that we wanted to prove. In the special case $\mathrm{fs}(R/xR) = 0$, we have that R/xR is F-rational, local, and hence a domain. But $\mathrm{ht}(J/xR) = 0$ and so $J = xR$ which is a contradiction ($d \in J \setminus xR$). Hence, in this case, $I(z)$ contains only the zero element, and $0 = \mathrm{ht}(I(z)) = \mathrm{fs}(R)$. □

Proposition 1.10. *Let R be a reduced ring. Let $I \subseteq R$ be an ideal such that $I^{[q]}$ is contracted with respect to the Frobenius map for every q. Then, for any x in the tight closure of I, $I(x)$ is a radical ideal. In particular, if R is F-injective, then the ideal $I(x)$ is radical for every $x \notin I$.*

Proof. For a proof see Proposition 2.5, [2].

□

Let us recall briefly the notion of F-unstability introduced by Fedder and Watanabe.

Definition 1.11. Suppose R has dimension d and let S denote the socle of $H^d_m(R)$. $F^e(S)$ will simply denote the set $\{s^{p^e} : s \in S\}$. We say that R is *F-stable* if $S \cap F^e(S) \neq 0$ for infinitely many e. Let us say that R is *F-unstable* if it is not F-stable.

Proposition 1.12. (**Fedder-Watanabe**) *Let R be an F-injective ring. Then R is F-stable if and only if there exists $0 \neq \eta \in S$ such that $F^e(\eta) \in S$ for every $e \geq 0$.*

Corollary 1.13. *Let R be an F-injective ring. Then R is F-stable if and only if* $\mathrm{fs}(R) = \dim(R)$.

Proof. Indeed, R is F-stable if and only if there exists $0 \neq \eta \in S$ such that $F^e(\eta) \in S$ for every $e \gg 0$. Therefore, by Lemma 1.4 R is F-stable if and only if there exists some $x \notin I$ such that $I(x) = m$. Since R is F-injective, the last assertion is equivalent to saying that $I(x)$ is a radical ideal of height equal to $\dim(R)$, i.e. $\mathrm{fs}(R) = \dim(R)$. □

Now the following theorem of Fedder and Watanabe is a natural consequence of the work done so far.

Theorem 1.14. *Let R be a Cohen-Macaulay ring which is F-finite and has an isolated singularity. R is F-rational if and only if R is F-injective and F-unstable.*

Proof. Once we know that R is a F-finite ring with an isolated singularity, we conclude from Corollary 1.8 that $\mathrm{fs}(R) \in \{0, \dim(R), \infty\}$. Now the above Corollary together with Remark 1.6 ends the proof. □

Remark 1.15. The introduction of the F-stability number puts in perspective the previous result. For F-injective rings that have non-isolated singularities, F-rationality ($0 = \mathrm{fs}(R)$) is not necessarily equivalent to F-unstability anymore ($\mathrm{fs}(R) < \dim(R)$).

2 F-stable primes

Let us continue by discussing an interesting property of the set of ideals that defines the F-stability number of R. Fix an ideal I of R that is generated by a system of parameters. Denote by $Z_{I,R}$ the set of ideals $I(x)$ where $x \notin I$. It follows from Proposition 1.10 that $Z_{I,R}$ contains only radical ideals when R is F-injective.

Theorem 2.1. *If R is F-injective and Cohen-Macaulay, then* $\mathrm{Max}(Z_{I,R})$ *is independent of the choice of I, finite and contains only prime ideals.*

Proof. We will show first that any ideal in $Z_{I,R}$ maximal with respect to inclusion is a prime ideal. Let us consider such an ideal $P = I(x) = \cap_{q=p^e \gg 0}(I^{[q]} : x^q)$, where $x \notin I$. In fact, when R if F-injective, we can show that $P = I(x) = \cap_{q=p^e \geq 1}(I^{[q]} : x^q)$. Indeed, if not, then we can consider an element c in P and $q_0 \geq p$ minimal with the property that $cx^{q_0} \in I^{[q_0]}$. But this implies that $c^p x^{q_0} \in I^{[q_0]}$. R is F-injective, so we can take p^{th} roots and get $cx^{q_0/p} \in I^{[q_0/p]}$. This contradicts the minimality of q_0.

Now, let us show that P is a prime ideal. Let $a, b \in R$ such that $ab \in P$ and suppose that $a \notin P$. Let us assume that b is not in P either. We will reach a contradiction. If $b \notin P$ then there is q such that $bx^q \notin I^{[q]}$. It is easy to see that a belongs to $I(bx)$. If this ideal is proper, because $I(x) \subset I(bx)$ it must equal $I(x)$, and then $a \in P$, which is false. Hence, $I(bx) = R$ and this means that $bx \in I$. Fix q and denote the ideal $I^{[q]}$ by J. This is also a parameter ideal. It may be easily seen that $a \in J(bx^q)$. Now, $I(x) \subset J(bx^q)$ and again either $I(x) = J(bx^q)$ or $J(bx^q) = R$ (here, we use the fact that definition of $\mathrm{Max}(Z_{I,R})$ is independent of the choice of the parameter ideal, and this follows from 1.4). In the first case, we get that $a \in P$ which is false. In the second case, we get $bx^q \in I^{[q]}$ which is a contradiction.

We will show now that $\mathrm{Max}(Z_{I,R})$ is finite. Denote by $I(x_m)$, where $m \in \Lambda$ the set of maximal ideals of Z. Here, Λ is a set of indices that will be shown to be finite. Consider the ideal generated by all the x_m's . Since R is Noetherian, this ideal is finitely generated, let's say by $x_1, ..., x_k$. Now, take $P = I(x)$ a maximal ideal of $Z_{I,R}$. Then, $x = a_1x_1 + ... + a_nx_n$. So $\cap_{i=1,\dots,k} I(x_i) \subset I(x)$. But $I(x)$ is prime, so there is $1 \leq h \leq k$ such that $I(x_h) \subset I(x)$. And since all these ideals are maximal in $Z_{I,R}$, we get equality. Therefore, every ideal of $\mathrm{Max}(Z_{I,R})$ is one of the ideals $I(x_i)$, where $i = 1...k$ and we are done. □

Remark 2.2. We will call these ideals *F-stable primes* and denote this set by $\mathrm{FS}(R)$ (see Corollary 2.8 for a motivation for this definition).

We have noted earlier that the parameter test ideal of R, $\tau_p(R)$, is contained in $I(x)$, for every $x \in I^*$. Therefore $\mathrm{ht}(\tau_p(R)) \leq \mathrm{fs}(R)$. In general, one does not have equality. This can be seen by considering the following example (suggested by Karen E. Smith):

Example 2.3. *Consider* (R, m, K) *and* (S, n, K) *two local Gorenstein* K*-algebras, where* K *is a field of characteristic* p. *Let* $A = (R \otimes_K S)_{m \otimes S + R \otimes n}$ *and assume that* $\mathrm{fs}(R) \neq \mathrm{fs}(S)$. *Then* $\mathrm{fs}(A) \neq \mathrm{ht}(\tau_p(A))$.

Proof. Let $x_1, ..., x_h$ be part of a full system of parameters in A and denote by J the ideal they generate. For $x \in J^* \setminus J$, one can extend the Definition 1.1 and set $J(x) := \{c \in A | cx^q \in J^{[q]}$, where $q = p^e$, for all $e \gg 0\}$. If $x_1, ..., x_h, ..., x_d$ form a full system of parameters for A, let $J_t = (x_1, ..., x_h, x_{h+1}^t, ..., x_d^t)$ for every $t > 0$. Then $x \in J_t^* \setminus J_t$ for some $t > 0$. So, $J(x) \subset J_t(x)$. (These considerations apply in fact to any local reduced Cohen-Macaulay ring A of characteristic $p > 0$.) Now, if $\mathrm{fs}(A) = \mathrm{ht}(\tau_p(A)$, then for every ideal J generated by a part of a system of parameters one has that $\mathrm{ht}(J(x)) = \mathrm{fs}(A)$, for every $x \in J^* \setminus J$. This holds because on one hand, $\tau_p(A) \subset J(x)$, and on the other hand $J(x) \subset P$, where P is some prime ideal belonging to $\mathrm{FS}(A)$. Thus, to prove our assertion is enough to find two ideals generated by parts of systems of parameters, say J' and J'', and two elements x', x'' such that the heights of $J'(x')$ and $J''(x'')$

are different. To construct these two ideals, one needs to consider an ideal I' in R generated by a system of parameters for R. and $x' \in R$ such that $\mathrm{fs}(R) = \mathrm{ht}(I'(x'))$, and similarly an ideal I'' in S generated by a system of parameters for S and $x'' \in R$ such that $\mathrm{fs}(S) = \mathrm{ht}(I''(x''))$. Now, take the images of these ideals in A and denote them by J' and J'' respectively. These ideals are generated by parts of a system of parameters for A. Clearly, $I'(x')A = J'(x')$ and $I''(x'')A = J''(x'')$ and the heights of $J'(x')$ and $J''(x'')$ are different being equal to $\mathrm{fs}(R)$ and $\mathrm{fs}(S)$, respectively. $\square$

In what follows we will give a characterization of the set $\mathrm{FS}(R)$ in terms of the prime ideals of R having the property that R_P is an F-stable ring. Before doing this, we need to reming briefly the concept of "co-localization" to the reader. We will assume that the ring R is complete and this implies that R admits a canonical module ω_R. The "co-localization" functor with respect to the prime ideal P associates, to each submodule of $H^d_m(R)$, a submodule of $H^k_P(R_P)$, where $k = \mathrm{ht}(P)$. This functor is "$(-)^{\vee_m^{\vee_P}}$" a double Matlis dual. The symbol "$(-)^{\vee_m}$" denotes the dual $\mathrm{Hom}_R(-, E(R/m))$, where $E(R/m)$ is the injective hull of R/m, whereas "$(-)^{\vee_P}$" denotes the dual $\mathrm{Hom}_R(-, E(R/P))$, where $E(R/P)$ is the injective hull of the R-module R/P. If P is omitted, then it is assumed to be m.

Let us remind the reader a few properties of this functor, proven by Karen E. Smith (see Lemma 2.1, [6]).

Proposition 2.4. *Let (R, m) be a Cohen-Macaulay complete local ring. Let N be a submodule of $H^d_m(R)$, where d is the dimension of R, and let W be any submodule of ω_R.*

i) $N^\vee \backsimeq \frac{\omega_R}{\mathrm{Ann}_{\omega_R} N}$;

ii) If $M \subset N$, then $(\frac{N}{M})^\vee \backsimeq \frac{\mathrm{Ann}_{\omega_R} M}{\mathrm{Ann}_{\omega_R} N}$;

iii) $W = \mathrm{Ann}_{\omega_R} N$ if and only if $\mathrm{Ann}_{H^d_m(R)} W = N$;

iv) If $W = \mathrm{Ann}_{\omega_R} N$, then

$$N^{\vee^{\vee_P}} = \mathrm{Ann}_{H^k_P(R_P)}(W \otimes R_P),$$

where k is the height of P.

There is another result which is very useful (Lemma 5.1, [6]). Before stating it, we recall that a submodule M of the highest local cohomology module of a ring is called F-stable if it is stable under the Frobenius action naturally induced on the local cohomology.

Proposition 2.5. *Let (R, m) be a Cohen-Macaulay complete local ring. If $N \subset H^d_m(R)$ is F-stable, then $N^{\vee^{\vee_P}}$ is also F-stable.*

Before we state the following result, we would like to introduce a notation. We denote by $\mathrm{F}-\mathrm{ann}(\eta)$ the set of elements $c \in R$ such that $cF^e(\eta) = 0$ for every e.

Proposition 2.6. *Let R be an F-injective Cohen-Macaulay complete local ring. Let I be an ideal generated by a system of parameters. Let P be a prime ideal in $Z_{I,R}$. Then R_P is F-stable.*

Proof. Write $P = I(x)$ and take $\eta = \mathrm{im}(x) \in H^d_m(R)$ under the map $R/I \longrightarrow H^d_m(R) = \varinjlim R/(x_1^t, ..., x_n^t)$. Then $\mathrm{F}-\mathrm{ann}(\eta)$ equals the prime P.

Let us denote by N the R-submodule of $H^d_m(R)$ generated by all the elements $F^e(\eta)$, $e \geq 0$. Obviously, N is F-stable which implies that $N^{\vee\vee_P}$ is also F-stable. Its annihilator in R_P equals $\mathrm{Ann}(N)R_P = \mathrm{F}-\mathrm{ann}(\eta)R_P = PR_P$. Hence, there exists $\gamma \neq 0 \in N^{\vee\vee_P}$. Consider now N_γ = F-stable submodule of $H^k_{PR_P}(R_P)$ generated by all the elements $F^e(\gamma)$, where $k = \mathrm{ht}(P)$. Its annihilator in R_P contains the annihilator of $N^{\vee\vee_P}$ and this is equal to PR_P. Therefore $\mathrm{Ann}(N_\gamma) = PR_P$ and that shows that $\mathrm{fs}(R_P) = \mathrm{ht}(P)$, so R_P is F-stable by Corollary 1.13.

□

Proposition 2.7. *Let R be an F-injective Cohen-Macaulay complete local ring. Let P be a prime ideal such that it is maximal with the property that R_P is F-stable. Then $P \in \mathrm{FS}(R)$.*

Proof. Since R_P is F-stable we can find an element $u \in R_P$ and $x_1, ..., x_k$ parameters contained in P such that $PR_P = (x_1, ..., x_k)(u)$. Here, $k = \mathrm{ht}(P)$ and we can assume that $u \in R$. Also, $\frac{u}{1} \notin (x_1, ..., x_k)R_P$. Then

$$Pu^q \in (x_1^q, ..., x_k^q)$$

for every $q \geq 0$.

Choose $x_{k+1}, ..., x_n$ in R such that $x_1, ..., x_n$ form a regular sequence in R.

In fact, because

$$c(u/1)^q \in (x_1^q, ..., x_k^q)R_P$$

we get that there exists $s_c(q) \in R - P$ such that

$$cs_c(q)u^q \in (x_1^q, ..., x_k^q)R.$$

But this means that

$$cu^q \in \cup_{w \notin P}(x_1, ..., x_k)^{[q]} : w = (x_1, ..., x_k)^{[q]} : s^{(k+1)q},$$

for some $s \notin P$, where the last equality is a result by Hochster and Huneke, Lemma 4.6 in [4]. So, $cu^q s^{(k+1)q} \in (x_1, ..., x_k)^{[q]}$ and this gives that $c \in (x_1, ..., x_k)(us^{k+1})$.

In conclusion, $c \in (x_1, ..., x_k, x_{k+1}^h, ..., x_n^h)(us^{k+1})$ for every h. So, $P \subset I(v)$, for some v, hence there is $Q \in \mathrm{Max}(Z)$ such that $P \subset Q$. But R_Q is F-stable, therefore $P = Q$. □

Putting these two Propositions together we have:

Corollary 2.8. *For an F-injective Cohen-Macaulay complete local ring R,* $\mathrm{FS}(R) = \mathrm{Max}\{P \in \mathrm{Spec}(R) : R_P \text{ is } F - \text{stable}\}$.

In general, it can be hard to compute the set of F-stable primes. But, in the Gorenstein case, the situation is simpler.

Observation 2.9. *Let (R, m, K) be an F-injective ring and not F-rational. Every F-stable prime P is of the form $P = I(u)$, where I is generated by a system of parameters and u belongs to the socle of R/I. In particular, if R is F-injective and Gorenstein, then* $\mathrm{FS}(R)$ *consists of a single prime ideal P. In this case, $P = I(u)$, where I is as above and u is the socle generator of R/I.*

Proof. Every element x in I^* has a nonzero multiple u in the $\mathrm{Soc}(R/I)$. That implies that $I(x) \subset I(u)$ and finishes the first part of the proof. It gives the second part also, as in the Gorenstein case the socle of R/I is one-dimensional over K. □

Example 2.10. 1. A local F-rational ring R has $\mathrm{fs}(R) = 0$.

2. If (R, m, K) is an F-injective isolated singularity which is not F-rational, then $\mathrm{fs}(R) = \dim(R)$. In this case, $\mathrm{FS}(R) = \{m\}$.

3. If (R, m, K) is F-injective Gorenstein, then $\mathrm{FS}(R)$ has only one element which equals m if and only if R is F-stable.

4. For non-isolated singularities the situation is much more complex and even the hypersurface case is interesting. In fact, our invariants could provide natural means of investigation of the Frobenius structure of these rings.

Let K be a field of characteristic p, where $p \equiv 1 \pmod 3$. Let

$$R = K[[x, y, z, t]]/(x^3 + y^3 + z^3).$$

This is an F-injective ring with a non-isolated singularity. We have that $\dim(R) = 3$, $\mathrm{FS}(R) = \{(x, y, z)\}$ and $\mathrm{fs}(R) = 2$. Therefore, this is an F-unstable non-isolated hypersurface singularity.

Now, let K be a perfect field of characteristic 2. Let

$$R = K[[x, y, z]]/(xyz + y^3 + z^3).$$

This is a non-isolated singularity. It can be checked that $\mathrm{FS}(R) = \{(x, y, z)\}$ and $\mathrm{fs}(R) = 2$. Hence, R is an F-stable non-isolated hypersurface singularity.

Each of the ideals $I(x)$, $x \in I^* - I$, contains the parameter test ideal, $\tau_p(R)$. Our study considers the maximal ideals with respect to the inclusion that have this form. If one is interested in computing the test parameter ideal or even its height, the set of ideals of the form $I(x)$ that are minimal with respect to the inclusion can be useful. More precisely, there are a couple of questions that can be asked for F-injective rings: Is there any parameter ideal I and $x \in I^*$ such that $\tau_p(R) = I(x)$? And if not, is there such an ideal I that has the height equal to $\mathrm{ht}(\tau_p(R))$? A positive answer to any of these questions would shed even more light on the structure of F-injective rings.

Remark 2.11. Some of the ideas mentioned in this note could be applied to the study of R-modules M that admit an injective Frobenius action, i.e. an injective additive map $F : M \to M$ such that $F(rm) = r^p F(m)$, for all $r \in R$ and $m \in M$. As in the case of the highest local cohomology module, one can define $\mathrm{F} - \mathrm{ann}(m)$ for every $0 \neq m \in M$ and look at the prime ideals of R of this form with the aim of developing a primary decomposition theory. This point of view is examined in [1].

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