BRIANÇON-SKODA FOR NOETHERIAN FILTRATIONS

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Abstract. In this note the Briançon-Skoda theorem is extended to Noetherian filtrations of ideals in a regular ring. The method of proof couples the Lipman-Sathaye approach with results due to Rees.

Let \( A \) be a regular ring of dimension \( d \) and \( I \) an ideal of \( A \). Let \( \overline{I} \) denote the integral closure of \( I \). The Briançon-Skoda theorem asserts that if \( I \) is generated by \( l \) elements then \( \overline{I}^{n+l} \subseteq I^{n+1} \), for all nonnegative integers \( n \). Moreover, \( \overline{I}^{n+d} \subseteq I^{n+1} \) for all nonnegative integers \( n \). Both statements were proven by Lipman-Sathaye [4]. These results have generated a considerable number of papers in commutative algebra and algebraic geometry, for a general discussion see for example Chapter 13 in [7] and Chapter 9 in [3].

In this paper, our aim is to prove a Briançon-Skoda type theorem for Noetherian filtrations. Our treatment will follow classical arguments by Lipman and Sathaye, and, respectively, Rees.

Let \( \mathcal{F} = \{I_n\}_n \) be a filtration of ideals of \( A \): that is, \( I_0 = A \), \( I_{n+1} \subseteq I_n \), and \( I_nI_m \subseteq I_{n+m} \), for all nonnegative \( n, m \). For any nonnegative integer \( k \), let \( \mathcal{F}(k) = \{I_{n+k}\}_n \) (technically this is not a filtration according to the above definition since on the zeroth spot we have \( I_k \) and not \( A \), but this will not affect what follows). Also, given two filtrations \( \mathcal{F} = \{I_n\}_n \) and \( \mathcal{G} = \{J_n\}_n \), we write \( \mathcal{F} \subseteq \mathcal{G} \) if \( I_n \subseteq J_n \) for all \( n \).

The filtration \( \mathcal{F} \) is called Noetherian if its Rees algebra \( R = \oplus_{n \geq 0} I_nt^n \) is Noetherian over \( A \). This holds if and only if its extended Rees algebra \( S = R[t^{-1}] \subset A[t^{-1}, t] \) is Noetherian.

There are various definitions of Noetherian filtrations in the literature. We follow the terminology used by Rees in [6], although the reader should be aware that for sake of readability we will avoid the interpretation of filtrations as special real valued functions on the ring \( A \). What we call here Noetherian filtration is called in some papers an essentially power filtration, Definition (2.1.2) and Remark (2.2) in [1].

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We note that the filtration given by the powers of a single ideal in a Noetherian ring forms itself a Noetherian filtration.

Another characterization of Noetherian filtration is as follows: A filtration $\mathcal{F} = \{I_n\}_n$ is Noetherian if and only if there exists $m \geq 1$ such that for all $n$, $I_n = \sum I_1^{e_1} \cdots I_m^{e_m}$, where the sum ranges over all nonnegative integers $e_1, \ldots, e_m$ such that $e_1 + 2e_2 + \cdots + me_m = n$. This was proven by Ratliff, see [5] and [1], Remark (2.2).

The integral closure of $R$ in $A[t]$ is a $\mathbb{N}$-graded ring, $\overline{R} = \bigoplus_{n \geq 0} J_n t^n$. As in [6], the integral closure of $\mathcal{F}$ is then defined by $\overline{\mathcal{F}} = \{J_n\}_n$. This is equivalent to saying that $x \in J_n$ if and only if there exist elements $a_i \in I_{n_i}$ and a positive integer $m$ such that

$$x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m = 0.$$  

Some authors call the filtration $\mathcal{G} = \overline{\mathcal{F}} = \{J_n\}_n$ the integral closure of the filtration $\mathcal{F} = \{I_n\}_n$. It should be noted that $\mathcal{G} \subseteq \overline{\mathcal{F}}$ with our definition. Note that $J_n$ belongs to the radical of $I_n$, so the filtration and its integral closure share the same radical (the radical of a filtration is the radical of any of its components).

A reduction for $\mathcal{F} = \{I_n\}_n$ is a filtration $\mathcal{G} = \{L_n\}_n$ such that $\mathcal{G} \subseteq \mathcal{F}$ and with the same integral closure filtration. A reduction is called basic if its Rees algebra is generated over $A$ by the least number of elements.

Recently, Küronya and Wolfe have studied extensions of the Briançon-Skoda theorem to graded systems of ideals. A family of ideals of $A$, $a_* = (a_n)_n$ is called a graded system of ideals if $a_n a_m \subseteq a_{n+m}$ for all nonnegative $n, m$. One should note that the family of ideals is not assumed descending. Küronya and Wolfe established a Briançon-Skoda type theorem for a particular type of graded systems, named stable graded system of ideals, that arise in algebraic geometry. Their generalization states that for a stable graded system of ideals $a_*$, there exists a positive constant $C$ such that, for all $n \gg 0$, $a_{Cn} \subseteq a_n$ (see Corollary 3.4 in [2]). Our statement will be stronger than this but under different hypotheses. The authors obtain in fact a statement regarding multiplier ideals of an graded system of ideals, as defined in Lazarsfeld [3]. For details on this statement and the definition of stable graded systems of ideals we refer the reader to [2].

Given a filtration $\mathcal{F} = \{I_n\}_n$, we call $a_1 t^{k_1}, \ldots, a_h t^{k_h}$ a system of generators for $\mathcal{F}$ if $a_1 t^{k_1}, \ldots, a_h t^{k_h}, u = t^{-1}$ generate the extended Rees algebra $(\bigoplus_{n \geq 0} I_n t^n)[u] = A[u, I_n t^n : n \geq 0]$. It can be arranged that $a_i \in I_{k_i} \setminus I_{k_{i+1}}, i = 1, \ldots, h$. The numbers $k_i$ are referred to as degrees of the generators.

Before we state the main result, we need to introduce more notations.
For a filtration $G = \{L_n\}_n$, let $g_1 t^{k_1}, \ldots, g_h t^{k_h}$ be a minimal set of generators of $G$. Let $k := \text{the least common multiple of } k_1, \ldots, k_s = [k_1, \ldots, k_s]$.

**Theorem.** Let $A$ be a $d$-dimensional regular ring and let $F = \{I^n\}_n$ be a Noetherian filtration of ideals of $A$.

For a reduction of $F$, let $l$ denote the sum of the degrees of the generators of the reduction. Also, consider $G = \{L_n\}_n$ a basic reduction of $F$, and let $k$ defined as above corresponding to $G$.

Then $F(m) \subset F$, where $m := \min\{l - 1, kd - 1\}$.

It is clear that in the case $F = \{I^n\}_n$, with $I$ ideal of $A$, we obtain the standard Briançon-Skoda theorem since in this case $l$ can be taken to equal the number of generators of $I$ and $k = 1$.

In his Strong Valuation Theorem, Rees has already shown that there exist a positive integer $k$ such that $F(k) \subset F$ Theorem 5.33, [6]. However, the integer $k$ produced by this result depends upon the degrees of the generators of the integral closure $\overline{R}$ over $R$ which are hard to estimate even in the classical case of a filtration of the type $\{I^n\}_n$, for an ideal $I$ of $A$.

**Proof.** The proof of the theorem will follow closely the Lipman-Sathaye proof of the standard Briançon-Skoda theorem. We found the exposition in [7] particularly useful and we will follow it for the first part of the proof.

First we will show that $F(l - 1) \subset F$.

For a finitely generated extension of rings $A \subseteq B$, we will write $J_{B/A}$ for the Jacobian ideal of $B$ over $A$.

For a reduction $F' = \{I'_n\}$ of $F$, we have that $F' = \overline{F}$ and $F' \subseteq F$, so if we show that $F'(l - 1) \subseteq F'$, then this implies that $F(l - 1) \subseteq F' \subseteq F$.

We will work with the reduction $F'$ and, by relabeling, we will still call it $F$.

We can localize and assume hence that our regular ring $A$ is local.

For the Noetherian filtration $F$, call its minimal generators $f_1 t^{l_1}, \ldots, f_r t^{l_r}$, with $f_i \in I_{l_i}$.

Consider the extended Rees algebra of our filtration $S = A[u, I_n t^n : n \geq 0]$ where $u = t^{-1}$. Clearly we can rewrite $S = A[u, f_1 t^{l_1}, \ldots, f_r t^{l_r}] = A[u, f_1/u^{l_1}, \ldots, f_r/u^{l_r}]$. Now let $B = A[u]$. Since $S = B[f_1/u^{l_1}, \ldots, f_r/u^{l_r}]$, a standard argument allows us to conclude that $u^{l_1+\ldots+l_r} \in J_{S/B}$ (see Lemma 13.3.1 in [7]). Let $l = l_1 + \ldots + l_r$. 

Let \( \overline{S} \) be the integral closure of \( S \). Since \( S \) is finitely generated over \( A[u] = B \), \( B \) is regular and the fraction field of \( S \) is separable over \( B \), we get that \( \overline{S} \) is module finite over \( S \).

We need to apply the following important result

**Theorem** (Lipman-Sathaye, Theorem 2, [4] or Theorem 12.3.10, [7]).

Let \( R \) be a Cohen-Macaulay domain with field of fractions \( K \). Let \( S \) be a domain that is finitely generated \( R \)-algebra. Assume that the field of fractions of \( S \) is separable and finite over \( K \) and that the integral closure \( \overline{S} \) of \( S \) is a finitely generated \( S \)-module. Assume that for all prime ideal \( Q \) in \( S \) of height one, \( R_{Q,R} \) is a regular local ring. Then

\[
\overline{S} :_L \mathcal{J}_{S/R} \subset S :_L \mathcal{J}_{S/R}.
\]

In particular, \( \mathcal{J}_{S/R} \overline{S} \subset S \).

The fraction field of \( S \) is the fraction field of \( B \), so Lipman-Sathaye Theorem applied to \( B \) and \( S \) gives that \( \mathcal{J}_{S/B} \overline{S} \subset S \). In particular \( u^-1 \overline{S} \subset S \).

At this stage we need another difficult result by Lipman-Sathaye:

**Proposition** (Lipman-Sathaye, Lemma, [4] or Theorem 13.3.2, [7]).

Let \( R \) be a regular domain with field of fractions \( K \). Let \( L \) be finite separable field extension of \( K \) and \( S \) be a finitely generated \( R \)-algebra in \( L \) with integral closure \( T \). Let \( 0 \neq t \) be such that \( R/tR \) is regular. Then if \( ts \cap R \neq tR \), then \( \mathcal{J}_{T/R} \subset tT \).

We can check that \( I_1 \subset uS \cap B \), but not in \( uB \), and \( B/uB \) is regular.

The above Proposition applies and gives \( J_{S/B} \subset u\overline{S} \).

So, \( u^-1 \overline{S} \subset \overline{S} : J_{S/B} \subset S : J_{S/B} \) by 12.3.10 (\( \overline{S} \) is module finite over \( S \)) so \( u^-1 J_{S/B} \overline{S} \subset S \) which gives \( u^-1 \overline{S} \subset S \).

But \( \overline{S} = \oplus_n K_n t^n \) so \( K_n t^{n-l+1} \subset I_{n-l+1} \). In particular

\[
J_{n+l-1} \subset K_{n+l-1} \subset I_n,
\]

or

\[
\mathcal{F}(l-1) \subset \mathcal{F}.
\]

Now we will show that \( \mathcal{F}(kd-1) \subset \mathcal{F} \).

For every positive integer \( k \) and every ideal \( J \) of \( A \) one can define a filtration denoted \( kJ \) in the following way:

\[
(kJ)_n = J^{\lceil n/k \rceil},
\]

for all nonnegative \( n \).
We can localize at a prime ideal containing the radical of the filtration $\mathcal{F}$, we can assume that we are in the local case.

Rees has proved that given a Noetherian filtration $\mathcal{F} = \{I_n\}_{n \geq 0}$ there exists a positive integer $k$ and an ideal $J$ such that $\mathcal{F}$ and $kJ$ are equivalent, that is they have the same integral closure, Theorem 6.12 and its Corollary, [6].

In fact this number $k$ is obtained by Rees as described in the statement of the theorem. Referring to the notations introduced just above the theorem, one chooses first a basic reduction $\mathcal{G} = \{L_n\}_n$ for $\mathcal{F}$. For $r_i = k/k_i$, let $a_i = g_i^{r_i}$. The ideal $J$ mentioned in the paragraph above is $J = (a_1, \ldots, a_h)$ and moreover the filtration $kJ$ represents a basic reduction for $\mathcal{F}$.

As before, let us denote the integral closure of $\mathcal{F} = \{I_n\}_n$ by $\{J_n\}_n$, and hence, by the above paragraph, this also represents the integral closure of $kJ$.

According to the definition of the integral closure of a filtration, we see that an element $x$ belongs to $J_n$ if and only if there exist elements $a_i \in (kJ)_{ni}$ and a positive integer $m$ such that

$$x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m = 0.$$ 

Since $((kJ)_n)^i \subset (kJ)_{ni}$, it follows that $(kJ)_n \subset J_n$.

We would like to remark that

$$\left\lceil \frac{n-k+1}{k} \right\rceil i \leq \left\lceil \frac{ni}{k} \right\rceil :$$

this follows easily, since $\left\lceil \frac{n-k+1}{k} \right\rceil i = \left\lceil \frac{n+1}{k} \right\rceil i - i$ and $\left\lfloor \frac{n+1}{k} \right\rfloor i - i \leq \left\lceil \frac{ni}{k} \right\rceil$.

With this in mind we see that $(kJ)_{ni} \subset ((kJ)_{n-k+1})^i$ which implies that for $n \geq k$, $J_n \subset (kJ)_{n-k+1}$.

Now we are in position to apply Briançon-Skoda for ideals in regular rings of dimension $d$:

$$J_n \subset (kJ)_{n-k+1} = J^{\left\lceil \frac{n-k+1}{k} \right\rceil} \subseteq J^{\left\lceil \frac{n-k+1}{k} \right\rceil - (d-1)} = (kJ)_{n-kd+1} \subseteq I_{n-kd+1}.$$

Putting everything together,

$$\mathcal{F}(kd-1) \subset \mathcal{F}.$$

We would like to illustrate our result with an example.

Let $A = k[[x, y]]$ where $k$ is a field, $I_1 = (x, y^2), I_2 = (x^2, xy^2, y^3)$. Note that $I_1^2 \subset I_2 \subset I_1$. Define $I_n = \sum I_1^{e_1} I_2^{e_2}$, where sum ranges over
all nonnegative integers $e_1, e_2$ such that $e_1 + 2e_2 = n$. The filtration $\mathcal{F} = \{I_n\}_n$ is Noetherian and its extended Rees algebra

$$S = A[t^{-1}, I_n t^n : n \geq 0]$$

is generated by $xt, y^2t, y^3t^2$.

According to results by Rees, Theorem 6.12 and Lemma 6.13 in [6], we know that a basic reduction of $\mathcal{F}$ must have 2 generators. Note that $y^2t$ is integral over $S : (y^2t)^2 - y(y^3t^2) = 0$. So, if $S' = A[xt, y^3t^2, t^{-1}]$ then the integral closure of $S'$ is $S$. The generators are $xt, y^3t^2$ which live in degrees 1 and 2. Applying the Theorem, we get that $m = 2$, so $\mathcal{F}(2) \subseteq \mathcal{F}$.

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References