Homework-Assignment 1 **Name:** ____________________

Write-up your solution carefully including all the details of the proof. Due Wednesday September 2.

(1) (5 points)
Let A, B be two sets. Show that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Proof. Let $x \in A \setminus B$. The certainly $x \in A$ so $x \in A \cup B$ and since $x \notin B$, then certainly $x \notin A \cap B$ (since $A \cap B \subseteq B$). So, $A \setminus B \subseteq (A \cup B) \setminus (A \cap B)$.

Similarly, $B \setminus A \subseteq (A \cup B) \setminus (A \cap B)$.

Putting the two together we get $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$.

For the reverse inclusion let us take $x \in (A \cup B) \setminus (A \cap B)$.

Then $x \in A \cup B$ so $x \in A$ or $x \in B$. Without any loss of generality, we can assume that $x \in A$.

But $x \notin A \cap B$. So, $x \notin A$ or $x \notin B$ (this is crucial!). But we know that $x \in A$ so $x \notin B$.

In conclusion $x \in A \setminus B$ which is a subset of $(A \setminus B) \cup (B \setminus A)$. So, the reverse inclusion is proven, and this ends the problem. $\square$

(2) (5 points) Let a, b be two positive integers. Show that $ab = (a, b)[a, b]$.

(3) (5 points)
Show by using mathematical induction that

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

for all positive integers n.

Proof. Set P(n):

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

Verification:
P(1): $1^2 = 1(4-1)/3 = 1$.
Assume P(n). Let us prove P(n+1).
First note that P(n+1) is

$$1^2 + 3^2 + \cdots + (2n-1)^2 + (2n+1)^2 = \frac{(n+1)(4(n+1)^2-1)}{3}$$

So start with

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

and add $(2n+1)^2$ to both sides.

It remains to show that

$$\frac{n(4n^2-1)}{3} + (2n+1)^2 = \frac{(n+1)(4(n+1)^2-1)}{3},$$

whihc can be easily checked. $\square$

(4) (5 points)
Let A be a set and $f, g : A \to A$ two functions such that $f \circ g$ is onto. Prove that f is onto.

(5) (5 points) (for graduate students only)
Let A be a set with m elements and B a set with n elements. How many functions $f : A \to B$ are there? Fully explain.

Proof. To define a function $f : A \to B$, we need to assign to each element in A an element in B. There are n possible ways of doing this and there are m elements of A (hence m assignments).

So the number is

$$n \cdot \ldots \cdot n = n^m,$$

where the above product has m terms.

□