Homework-Assignment 2
Selected solutions.

(1) (5 points)
Let $z = 1 - \sqrt{3}i$. Compute $\mid z \mid, \overline{z}, z^{20}$.

(2) (5 points) Solve the equation

$$z^4 = i.$$

Proof. Let us find the polar coordinates for i. The absolute value of i is 1. The angle of i is $\pi/2$.
So, $i = e^{i\cdot(\pi/2+2k\pi)}$, where k is an integer.
Let $z = re^i\theta$. So,
$z^4 = r^4 e^{i4\theta} = e^{i\cdot(\pi/2+2k\pi)}$.
So, $r^4 = 1, 4\theta = \pi/2 + 2k\pi$.
Hence $r = 1, \theta = \pi/8 + k\pi/2$. The distinct solutions are obtained for $k = 0, 1, 2, 3$:

$$e^{i\cdot(\pi/8+k\pi/2)},$$

so $e^{i(\pi/8)}, e^{i(5\pi/8)}, e^{i\cdot(9\pi/8)}, e^{i\cdot(13\pi/8)}$ are the solutions. □

(3) (5 points)
Show that if $a \mid m$, $b \mid m$ and $(a, b) = 1$, then $ab \mid m$.

Proof. $a \mid m$ so $ab \mid bm$. Similarly $b \mid m$ so $ab \mid am$.
Since $(a, b) = 1$ there exist u, v integers such that $au + bv = 1$.
Since $au + bv = 1$ we get (after multiplying both sides by m):

$$amu + bmv = m.$$

We have $ab \mid bm$, so $ab \mid bmv$.
Also $ab \mid am$, so $ab \mid amu$.
Since $ab \mid amu, ab \mid bmv$, we get $ab \mid m$. □

(4) (5 points)
Let A be a finite set and $f : A \to A$ an injective function. Prove that f is onto.

Proof. Because f is injective, we have that if $x \neq y$ then $f(x) \neq f(y)$. Then $Im(f)$ has the same number of elements as A. But $Im(f) \subseteq A$. So, $Im(f) = A$ since they have the same number of elements.
But $Im(f) = A$ means that f is onto. □

(5) (5 points) (for graduate students only) A positive integer is called a square if it is the square of an integer. Prove that if a, b are squares and $(a, b) = 1$, then ab is a square.