Homework-Assignment 3
Selected solutions.

(1) (5 points)
Prove that is α is a cycle of length l, then $\alpha^l = 1$.

(2) (5 points) Find a complete factorization of

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 4 & 3 & 7 & 5 & 9 & 8 & 6 \end{pmatrix}$$

Compute the order of the permutation.

Proof. The permutation decomposes completely as :

$$(12)(34)(5796)(8).$$

The order equals $lcm(2, 2, 4) = 4$.

□

(3) (5 points) Let $n \geq 3$ and $\alpha \in S_n$ such that α commutes with every $\beta \in S_n$. Prove that $\alpha = e$.

Proof. Assume that α is not the identity so it contains a nontrivial cycle in its factorization.
Let us write $\alpha = \beta_1 \cdots \beta_s$ with β_i disjoint cycles, and $s \geq 1$.
Say $\beta_1 = (i \;\; j \;\; ...)$.
Since $n \geq 3$, there exists $k \neq i, k \neq j$.
Then $\alpha(j \;\; k) \neq (j \;\; k)\alpha$ because
the permutation on the left sends i to j while the one on the right sends i to k and $k \neq j$.
In conclusion, if α is not the identity, then we can find a permutation that does not commute with α, contradiction.
So, $\alpha = e$.

□

(4) (5 points)
Compute 4^{300} modulo 7.

Proof. $4^3 = 64 = 63 + 1$, so $4^3 \equiv 1$ modul0 7.
Therefore,
$4^{300} = (4^3)^{100} \equiv 1^{100} = 1$, modulo 7.

□

(5) (5 points)(for graduate students only)
If $1 < r \leq n$ prove that there are

$$\frac{n(n-1)\cdots(n-r+1)}{r}$$

cycles of length r in S_n.

Proof. The number of ordered subsets of r elements from $\{1, ..., n\}$ is $\frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1)$.
But a cycle $(i_1 i_2 ... i_r)$ is given by r such sets: $\{i_1, i_2, \ldots, i_r\}, \{i_2, i_3 \ldots, i_1\}$, $\{i_3, i_4, \ldots, i_1, i_2\}$ etc..
So, the answer is obtained by dividing to r

$$\frac{n(n-1)\cdots(n-r+1)}{r}.$$

□