

Homework-Assignment 4 Name: _____

Please staple your assignment. Write-up your solution carefully including all the details of the proof. Due Monday October 5.

(1) (5 points)

Check that $\mathbf{R} \setminus \{-1\}$ together with the operation $*$ defined by:

$$a * b := a + b + ab$$

is an abelian group.

Proof. First let us check that if $a, b \in G$ then $a * b \in G$. This means that we need to verify that $a, b \neq -1$ implies that $ab + a + b \neq -1$.

But $ab + a + b = -1$ is equivalent to $ab + a + b + 1 = 0$ which means that $(a + 1)(b + 1) = 0$ and hence $a = -1$ or $b = -1$, impossible. So, $a * b \in G$.

Associativity:

$$\begin{aligned} (a * b) * c &= (a * b)c + a * b + c = (ab + a + b)c + ab + a + b + c = abc + ac + bc + ab + a + b + c. \\ a * (b * c) &= a(b * c) + a + b * c = a(bc + b + c) + a + bc + b + c = abc + ab + ac + a + bc + b + c = \\ &= abc + ab + ac + bc + a + b + c. \end{aligned}$$

So, $(a * b) * c = a * (b * c)$.

The identity element:

We need $e \in G$ such that $a * e = e * a = a$.

But $a * e = e * a = ae + e + a$ so $ae + e + a = a$ implies $(a + 1)e = 0$ and so $e = 0$ since $a \neq -1$.

Inverse:

Let $a \neq -1$. We need b such that $a * b = ab + a + b = e$, and hence $ab + a + b = 0$. Solving for b we get $b = \frac{-a}{1+a}$.

Check that $b \neq -1$: if $\frac{-a}{1+a} = -1$ then $-a = -1 - a$ and so $0 = -1$. Impossible.

Abelian group:

$$a * b = ab + a + b = ba + b + a = b * a.$$

□

(2) (5 points) Let $\sigma \in S_n$ be a cycle of length l . Prove that if $\sigma = \tau_1 \cdots \tau_s$, where τ_i are transpositions, then $s \geq l - 1$.

(3) (5 points) Prove that in S_n there are as many even permutations as odd permutations.

Proof. Let $E = \{\sigma : \sigma \text{ even}\}$ and $O = \{\sigma : \sigma \text{ odd}\}$.

Let $F : E \rightarrow O$, $F(\sigma) = \sigma(12)$.

We will show that F is an 1-1 correspondence:

Say $F(\sigma) = F(\tau)$. So, $\sigma(12) = \tau(12)$. Hence $\sigma(12)(12)^{-1} = \tau(12)(12)^{-1}$ or $\sigma = \tau$. This shows that F is 1-1.

To show F is onto, consider $\alpha \in O$. We need $\sigma \in E$ such that $F(\sigma) = \alpha$.

Take $\sigma = \alpha(12)$. Since α is odd we get that $\alpha(12)$ is even so $\sigma \in E$.

Also, $F(\sigma) = F(\alpha(12)) = \alpha(12)(12) = \alpha$.

This finishes the proof.

□

(4) (5 points) Show that there is some positive integer k , such that $f^k = e$ for all $f \in S_n$, where e is the identity function. (k is INDEPENDENT of any f in S_n)

(5) (5 points)(for graduate students only)

A permutation of the form $\sigma = (i \ i + 1)$, $i < n$ is called adjacency.

Prove that every transposition $(i \ j)$ is a product of an odd number of adjacencies.

Proof. Let $j - i = n$ so $(i \ j) = (i \ i + n)$.

We will do induction on $n = j - i$. We will call this number the distance of the transposition.

$n = 1$: $(i \ i + 1)$ is already an adjacency.

Assume that any transposition of distance n is a product of adjacencies.

Then let $(i \ i + (n + 1))$ be a transposition of distance $n + 1$.

Note that

$$(i \ i + (n + 1)) = (i \ i + 1)(i + 1 \ i + (n + 1))(i \ i + 1).$$

The transposition $(i + 1 \ i + (n + 1))$ has distance $n + 1 - 1 = n$ so by the induction step is a product of adjacencies. So, $(i \ i + (n + 1))$ is a product of adjacencies as well.

Finally, any transposition is odd so the number of adjacencies that appears in its decomposition is odd as well.

□