Homework-Assignment 5
Selected solutions.

(1) (5 points) Let $*$ be an operation defined on a set G such that $*$ is associative.
Assume that there exists an element $e \in G$ such that $e * a = a$ for all $a \in G$. Also assume that for all $a \in G$, there exists $b \in G$ such that $b * a = e$.
Prove that $(G, *)$ is a group.

Proof. The operation is defined on G so it is closed. It is also associative.
We need to show $a * b = e$ and $a * e = a$.
Compute
$$(a * b)^2 = a * b * a * b = a * (b * a) * b = a * e * b = a * (e * b) = a * b.$$
For $a * b \in G$ there exists $c \in G$ such that $c * (a * b) = e$, by hypothesis.
So, $c * (a * b) * (a * b) = c * (a * b)$ or $e * (a * b) = e$ which gives $a * b = e$.
Finally
$$a * e = a * (b * a) = (a * b) * a = e * a = a.$$
□

(2) (5 points) Find all orders of the elements in $U(\mathbb{Z}_{12})$.

Proof. $U(\mathbb{Z}_{12}) = \{\overline{1}, \overline{5}, \overline{7}, \overline{11}\}$.
Then $ord(\overline{1}) = 1$.
Since $5^2 = 25 \equiv 1$ modulo 12 we get that $\overline{5}$ has order 2.
Similarly, $7^2 \equiv 11^2 \equiv 1$ modulo 12.
So $\overline{7}, \overline{11}$ also have order 2.
□

(3) (5 points)
Find all elements x in S_4 such that $x^4 = e$.

(4) (5 points) If G is a group in which $a^2 = e$ for all $a \in G$, then G is abelian.

(5) (5 points) (for graduate students only) Show that a group of order 4 must be abelian.

Proof. Let a, b two elements in a group G. Hence $G = \{e, a, b, c\}$.
Then ab is different than a and b so either $ab = e$ (in which case a and b are inverses to each other and hence commute) or $ab = c$.
Similarly, ba is different than a and b. Als, $ba = e$ implies that a and b are inverses to each other and hence commute. Therefore $ba = c$.
So $ab = ba = c$ so a, b commute.
This is true for every a, b in G so G is Abelian.
□