1. **Grace’s Theorem; Resultants**

In the proof of Grace’s theorem we will use the following Lemma.

**Lemma 1.1.** Let all roots $z_1, \ldots, z_n$ of $f(z)$ lie inside the circular domain $K$ and let $\xi$ lie outside $K$. Then all roots of $A_\xi f(z)$ lie inside $K$.

**Proof.** Assume first that $\xi \neq \infty$. Then $w$ is a root of $A_\xi f(z)$ then $\xi$ is the center of mass of the roots of $f$ with respect to $w$.

Theorem 1.6 from Lecture 6 says that since $z_1, \ldots, z_n$ are in $K$ and the center is not in $K$, then $w$ must be in $K$.

If $\xi = \infty$, then $A_\xi f(z) = f'(z)$. The condition that $\xi \notin K$ means that $K$ is not the exterior of a circle so $K$ is a convex set and hence it contains the convex hull of $z_1, \ldots, z_n$. Now, Gauss-Lucas Theorem shows that the critical points of $f(z)$ are in $K$ as well.

\[\square\]

We will prove now the last theorem stated in Lecture 9.

**Theorem 1.2** (J. H. Grace, 1902). Let $f, g$ be two apolar polynomials. If all the roots of $g$ belong to a circular domain $K$, then at least one of the roots of $f$ also belongs to $K$.

**Proof.** Suppose that all the roots $z_1, \ldots, z_n$ of $f$ lie outside $K$.

By Lemma proven above, $A_{z_n} g(z)$ has all roots inside $K$. Repeated applications of the Lemma show that $A_{z_1} \ldots A_{z_n} f(z)$ has all roots in $K$. But this last expression is a polynomial of degree 1 hence of the form $c(z - a)$, $c \neq 0$. So, $a \in K$.

Let us compute remember that $f, g$ are apolar so $0 = A_{z_1} \ldots A_{z_n} g(z) = A_{z_1} (c(z - a)) = c(z_1 - a)$.

Hence $z_1 = a$. But $z_1 \in K$, while $a \notin K$. Contradiction.

So, at least one $z_i$ is in $K$.

\[\square\]

The next topic we will examine is that of resultants. The resultant of two polynomials $f, g$ whether they have a common root or not.

Let $f(x) = a_0x^n + \cdots + a_n$, $g(x) = b_0x^m + \cdots b_m$ with $a_i, b_j \in \mathbb{C}$, $a_0 \neq 0, b_0 \neq 0$.

**Proposition 1.3.** The polynomials $f, g$ have a common root if and only if there exist nonzero complex polynomials $p, q$ of degrees at most $m - 1$, respectively $n - 1$ such that $pf = qg$.

**Proof.** If $f, g$ have a common root $a$, then $x - a$ divides both $f$ and $g$, so $f = q(x - a)$ and $g = p(x - a)$ with $q, p$ nonzero polynomials of degrees $n - 1, m - 1$ respectively.

But then $pf = qg$, as it can easily be checked.

For the converse, $pf = qg$ implies that if none of the linear factors of $g$ appear in $f$ then they all must appear in $p$ but this shows that $g$ divides $p$ and hence the degree of $p$ is at least $m$ which is a contradiction.

So, at least one of the linear factors that appear in the factorization of $g$ must appear in the factorization of $f$ which is equivalent to saying that $f, g$ have a common root.

\[\square\]
Let us pursue what the previous result tells us.

Write \( q(x) = u_0 x^{m-1} + \cdots + u_{m-1} \), \( p(x) = v_0 x^{n-1} + \cdots + v_{n-1} \), where the coefficients are complex numbers not all zero.

Now, if we look at the equality

\[ p(x)f(x) = q(x)g(x), \]

we note that if we identify the coefficients of the polynomial expression for the left and right hand terms we get that it is equivalent to finding \( u_1, \ldots, u_{m-1} \) not all zero, and \( v_0, \ldots, v_{n-1} \) not all zero such that they satisfy the following system of equations:

\[ a_0 u_0 - b_0 v_0 = 0 \]
\[ a_1 u_0 + a_0 u_1 - b_1 v_0 - b_0 v_1 = 0 \]
\[ a_2 u_0 + a_1 u_1 + a_2 u_2 - b_2 v_0 - b_1 v_1 - b_0 v_2 = 0 \]

\[ \vdots \]

Note that the system has \( m + n \) lines and \( m + n \) unknown and it is homogeneous and linear. Such a system admits a nonzero solution (our \( u \)'s and \( v \)'s) if and only if its determinant is zero.

After rearranging the rows and columns we get the following determinant denoted by \( R(f, g) \).

The matrix that gives this determinant is called the Sylvester matrix of \( f, g \) (it is of size \( n + m \) by \( n + m \)).

\[
R(f, g) = \begin{vmatrix}
  a_0 & a_1 & \cdots & a_{n-1} & a_n & 0 & \cdots \\
  0 & a_0 & \cdots & a_{n-1} & a_n & 0 & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
  b_0 & b_1 & \cdots & b_{m-1} & b_m & 0 & \cdots \\
  0 & b_0 & \cdots & b_{m-1} & b_m & 0 & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots 
\end{vmatrix}
\]

Note that \( n, m \) are different that in fact the reader should not infer from the above presentation that \( a_n \) and \( b_m \) lie on same column. Also, the first \( m \) rows contain only \( a \)'s and the remaining \( n \) rows contain only \( b \)'s.

Hence we can conclude

**Theorem 1.4.** *The polynomials \( f, g \) share a root if and only if \( R(f, g) \neq 0 \).*

There is a more definite relation between the roots of \( f, g \) and the resultant.

**Theorem 1.5.** Let \( x_1, \ldots, x_n, y_1, \ldots, y_m \) be the roots of \( f \) respectively \( g \).

Then \( R(f, g) = a_0^m b_0^n \Pi(x_i - y_j) = a_0^m \Pi_{i=1}^n g(x_i) = (-1)^{nm} b_0^m \Pi_{j=1}^m f(y_j) \).

**Proof.** First let us notice that \( R(f, g) \) is a homogeneous polynomial of degree \( m \) in \( a_0, \ldots, a_n \) and degree \( n \) in \( b_0, \ldots, b_m \).

Since \( a_i/a_0, b_j/b_0 \) are symmetric expressions in \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_m \) respectively by Viète, we get that \( R(f, g) = a_0^m b_0^n P(x_1, \ldots, x_n, y_1, \ldots, y_m) \) with \( P \) a symmetric polynomial that vanishes whenever \( x_i = y_j \) for some \( i \) and \( j \).
But we can always use repeatedly the equality
\[ x_i^k = (x_i - y_j)x_i^{k-1} + x_i^{k-1}y_j \]
to write
\[ P(x_1, \ldots, x_n, y_1, \ldots, y_m) = (x_i - y_j)Q(x_1, \ldots, x_n, y_1, \ldots, y_m) + U(x_1, \ldots, x_i, \ldots, y_m). \]
where \( x_i \) symbolizes that \( x_i \) does not appear in \( U \).

But letting \( x_i = y_j \) makes \( R(f, g) = 0 \) so in fact \( U \equiv 0 \), and this shows that \( P \) must be divisible by all \( x_i - y_j \) and so \( R(f, g) \) is divisible by \( S = a_0^m b_0^n \Pi(x_i - y_j) \).

Write \( g(x) = b_0 \Pi(x - y_j). \) Hence \( S = a_0^m b_0^n g(x_i) \). Similarly, \( f(y_j) = (y_j - x_1) \cdots (y_j - x_n) = (-1)^n (x_1 - y_j) \cdots (x_n - y_j) \).

So, \( S = (-1)^{mn} \Pi_{j=1}^m f(y_j) \).

Now look at
\[ S = a_0^m \Pi_{i=1}^n (b_0 x_i^m + \cdots b_m). \]

This is a polynomial expression that has degree exactly \( n \) in \( b_0, \ldots, b_m \) (and homogenous). If we expand we see that \( S \) is a symmetric polynomial in \( x_1, \ldots, x_n \) and in fact by using Viète’s relations we see that \( S \) is homogenous of degree \( m \) is \( a_0, \ldots, a_n \). But \( S \) divides \( R(f, g) \) as polynomials so there must exist a constant \( \lambda \) such that
\[ R(f, g) = \lambda S. \]

By looking at the coefficient of \( x_1^m \cdots x_n^m \) in both sides we get that \( \lambda = 1 \). This finishes our proof. \qed

**Corollary 1.6.** \( R(g, f) = (-1)^{\deg(f)\deg(g)} R(f, g) \)

**Corollary 1.7.** If \( f = qg + r \), then
\[ R(f, g) = b_0^{\deg(f) - \deg(r)} R(r, g). \]

**Proof.** Let \( y_j \) be the roots of \( g \). Then \( f(y_j) = r(y_j) \), since \( g(y_j) = 0 \).

So, \( R(f, g) = b_0^{\deg(f)} \pi f(y_j) = b_0^{\deg(f) - \deg(r)} \pi r(y_j) = b_0^{\deg(f) - \deg(r)} R(r, g). \) \qed

**Corollary 1.8.**
\[ R(f, g) = R(f, g) R(f, h). \]

**Definition 1.9.** Let \( x_1, \ldots, x_n \) be the roots of a degree \( n \) polynomial \( f(x) = a_0 x^n + \cdots + a_n \).

The **discriminant** of \( f \) is
\[ D(f) = a_0^{n-2} \Pi_{i<j} (x_i - x_j)^2. \]

**Theorem 1.10.**
\[ R(f, f') = (-1)^{n(n-1)/2} a_0 D(f). \]
Proof. First note that $R(f, f') = a_0^{n-1}\Pi f'(x_i)$.
However, $f'(x) = a_0 \sum_{i=1}^{n}(x - x_1) \cdots (x - x_i) \cdots (x - x_n)$, so $f'(x_i) = a_0 \Pi_{i \neq j} (x_i - x_j)$.
Therefore

$$R(f, f') = a_0^{2n-1}\Pi_{i \neq j}(x_i - x_j) = (-1)^{n(n-1)/2} a_0^{2n-1}\Pi_{i < j}(x_i - x_j)^2.$$ \hfill \qed

Corollary 1.11. Let $f, g, h$ be monic polynomials. Then

$$D(fg) = D(f)D(g)R^2(f, g)$$

$$D(fgh) = D(f)D(g)D(h)R^2(f, g)R^2(g, h)R^2(h, f)$$

Theorem 1.12. Let $f$ be a real polynomial of degree $n$ without any real roots.

Then the sign of $D(f)$ is the sign of $(-1)^{n/2}$.

Proof. Let $f(x) = a_0(x - x_1) \cdots (x - x_n)$.

We can verify that $D((x - a)f(x)) = D(f(x))[f(a)^2]$.

Now, let $a, \overline{a}$ be conjugate roots of $f$: $f(x) = (x - a)(x - \overline{a})g(x)$.

Then

$$D(f) = D(g)(a - \overline{a})(g(a)g(\overline{a})^2),$$

which shows that $D(f) = -D(g)$. Continuing like this until we exhaust all roots of $f$ and we get our statement. \hfill \qed