1. Stable polynomials. Gauss-Lucas Theorem

We continue by proving in detail the theorem that closed Lecture 7.

**Theorem 1.1.** With the notations just introduced, if \( f \) has real coefficients then \( f \) is stable (i.e., all roots have negative real part) if and only if \( f \) and \( g \) have positive coefficients.

**Proof.** Suppose that \( f \) is stable. Then it can easily be shown that \( f, g \) have real coefficients:

Since \( f(z) = (z - z_1) \cdots (z - z_n) \), then if \( z_i \) is real and negative that the factor \( z - z_i \) has positive coefficients. If \( z_i \) is complex, not real, then its conjugate \( \overline{z} \) is a root as well (check this), so \( f \) has \( (z - z_i)(z - \overline{z}) \) as a factor. But \( (z - z_i)(z - \overline{z}) = z^2 - 2Re(z_i) + |z_i|^2 \) has only positive coefficients, as \( Re(z_i) < 0 \).

In conclusion, \( f \) is the product of polynomials with positive coefficients, so it has positive coefficients as well.

We can repeat the argument for \( g \), since \( g \) is also a stable polynomial as its roots are sums of roots of \( f \), which means that they will have negative real parts as well. We need to make sure that \( g \) has real coefficients as well. For every root of \( f \), its conjugate is also a root. Hence for every root of \( g \), say \( z_i + z_j \), its conjugate \( \overline{z_i} + \overline{z_j} = z_i + z_j \) is also a root of \( g \). Hence we can pair up a complex nonreal root \( w \) of \( g \) with its conjugate and note that \( g \) must be a product of terms of the form \( (z - w)(z - \overline{w}) = z^2 - 2Re(w)z + |w|^2 \), which have real coefficients, and terms of the form \( z - \alpha \), \( \alpha \) real, which also have real positive coefficients.

For the converse, if a polynomial has positive coefficients then it is clear that its real roots must be negative. This shows that \( f \) has negative real roots. For a complex root of \( f \), \( z = a + ib \), we see that \( \overline{z} = a - ib \) is a real root as well, so \( z + \overline{z} = 2a \) must be a REAL root of \( g \). But \( g \) has positive coefficients so \( 2a \) must be negative, so \( a \) is negative, hence the real part of \( z \) is negative. This shows that \( f \) is stable.

\[ \square \]

**Example 1.2.** Let \( f = z^2 + z + 2 \). Let us compute \( g \). It has degree 1 and root \( z_1 + z_2 \) where \( z_1, z_2 \) are roots of \( f \).

Note that Viète’s relations tell us that \( z_1 + z_2 = -1 \), so \( g(z) = z - (-1) = z + 1 \).

As we can see the above Theorem applies and \( f \) is stable.

In fact, one should notice that the coefficients of \( g \) are symmetric polynomials in \( z_1, ..., z_n \). Therefore the coefficients of \( g \) become polynomials in the coefficients of \( f \), after using the Viète’s relations.

Now, let us go back to the equation

\[ P(z)y'' + Q(z)y' + R(z)y = 0, \]

where \( P, Q, R \) are polynomials.

The following result was stated in lecture 7 without a proof, so we will provide a proof now.

But first, let us revisit the concept of multiplicity of a root for a polynomial.

**Proposition 1.3.** Let \( f(z) \) be a polynomial. Then \( z_0 \) is a root of multiplicity \( k \) if and only if \( f^{(i)}(z_0) = 0 \) for \( i \leq k - 1 \) and \( f^{(k)}(z_0) \neq 0 \), where \( f^{(i)}(z) \) stands for the \( k \)th order derivative of \( f \).
Proof. Let \( g(z) = f(z + z_0) \). Note that \( g(0) = 0 \).

Also, \( g^{(i)}(z) = f^{(i)}(z + z_0) \).

Moreover \( f(z) = (z - z_0)^k h(z) \) is equivalent to \( g(z) = z^k h(z + z_0) \) and of course \( h(z_0) \neq 0 \) is equivalent to \( h(0 + z_0) \neq 0 \). This says that 0 is a root of multiplicity \( k \) for \( g \) if and only if \( z_0 \) is root of multiplicity \( k \) for \( f \).

First let us assume that \( z_0 \) is root of multiplicity \( k \) for \( f \). Hence as we have see above, 0 is root of multiplicity \( k \) for \( g \) and \( g(z) = z^k p(z) \) where \( p \) is such that \( p(0) \neq 0 \).

So, \( g(z) = az^k + \ldots, a \neq 0 \) and it can be easily checked that \( g^{(i)}(0) = 0 \) for \( i \leq k \), and \( g^{(k+1)}(0) \neq 0 \). As remarked before, this is equivalent to \( f^{(i)}(z_0) = 0 \) for \( i \leq k-1 \) and \( f^{(k)}(z_0) \neq 0 \).

Now, let us assume that \( f^{(i)}(z_0) = 0 \) for \( i \leq k-1 \) and \( f^{(k)}(z_0) \neq 0 \), that is \( g^{(i)}(0) = 0 \) for \( i \leq k - 1 \), and \( g^{(k)}(0) \neq 0 \).

Let \( g(z) = a_0 + a_1 z + \ldots \).

But \( g(0) = 0 \) implies \( a_0 = 0 \). \( g'(0) = a_1 \) so this means that \( a_1 = 0 \). Similarly, \( g''(0) = 2a_2 \) and hence \( a_2 = 0 \).

Note that \( g^{(k)}(0) = k!a_k \), so \( a_k \neq 0 \).

So, we can write \( g(z) = a_k z^k + \ldots = z^k p(z) \), where \( p \) is a polynomial such that \( p(0) \neq 0 \). Hence \( g \) has 0 as a root of multiplicity \( k \), and therefore \( z_0 \) is root of multiplicity \( k \) for \( f \).

\[ \square \]

**Proposition 1.4.** The polynomial solutions of

\[ P(z)y'' + Q(z)y' + R(z)y = 0, \]

have only simple zeroes.

**Proof.** Assume that \( z_0 \) is a multiple zeroes for \( y \). Let us say that its multiplicity is \( k > 1 \).

Case 1: \( P(z_0) \neq 0 \).

Then by taking the derivative of

\[ P(z)y'' + Q(z)y' + R(z)y = 0, \]

\( k - 2 \) times we get \( P(z)y^{(k)}(z) + F(z) = 0 \) where \( F \) is an expression in the derivatives of \( y \) of order less or equal to \( k - 1 \). (If \( k = 1 \), there not need to take derivatives). When we plug in \( z = z_0 \) we get \( P(z_0)y^{(k)}(z_0) = 0 \) so \( y(z_0) = 0 \) which contradicts the fact \( z_0 \) has multiplicity exactly \( k \).

Case 2: \( P(z_0) = 0 \).

Take the derivative of

\[ P(z)y'' + Q(z)y' + R(z)y = 0, \]

and get

\[ P'(z)y'' + Py'''(z) + Q(z)y'' + Q'(z)y' + R'(z)y + R(z)y' = 0. \]

Now, remark that \( R' \equiv 0 \) (since \( R \) is a constant) and \( P \) has only simple roots, for \( P, R \) polynomials defining the Hermite, Laguerre, Legendre polynomials.

We either have \( P'(z_0) + Q(z_0) \neq 0 \), or \( Q'(z_0) + R' \neq 0 \). As in case 1, take the \( k - 2 \) or \( k - 1 \) derivatives of the newly found expression and note that one gets

\[ H(z)y^{(k)}(z) + F = 0 \]
where $F$ is an expression depending upon $P$ and the derivatives of $y$ of order less or equal to $k - 1$ such that $y^{(i)} = 0, \forall i \leq k - 1$ implies that $F = 0$. Here $H(z_0) \neq 0$. (Again, as before, there is no need to take derivatives if $k = 1$.)

When we plug in $z = z_0$ we see that $y^{(i)}(z_0) = 0$, for all $i \leq k - 1$, so we get $H(z_0)y^{(k)}(z_0) = 0$, as $F(z_0)$ vanishes. So, $y^{(k)}(z_0) = 0$, contradicting again the fact that the multiplicity of $y$ is $k$.

Hence $k = 1$ is the only possibility so $z_0$ is a simple root for $y$. 

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