Fall 2010 Polynomials Homework-Assignment 1

Solutions.

(1) (5 points) Consider the equation $ax^2 + bx + c = 0$ with real coefficients and no real solutions. Assume that $a + b + c > 0$. What sign does the expression $a - b + c$ have? Prove it.

Proof. Let $f(x) = ax^2 + bx + c$. The equation $f(x) = 0$ has no real solution and it is quadratic. This implies that it either $f(x) > 0, \forall x \in \mathbb{R}$ or $f(x) < 0, \forall x \in \mathbb{R}$. But $f(1) = a + b + c > 0$, so we must be in the first case: $f(x) > 0, \forall x \in \mathbb{R}$. In particular $f(-1) > 0$ and note that $f(-1) = a - b + c$. So $a - b + c > 0$.

(2) (5 points) If $\alpha$ and $\beta$ are the roots of $ax^2 + bx + c = 0$, write down the equation of degree two (and coefficients expressed in terms of $a, b, c$) that has roots $2\alpha + 3\beta$ and $2\beta + 3\alpha$.

Proof. Let $x_1 = 2\alpha + 3\beta, x_2 = 2\beta + 3\alpha$. We will compute the sum and product of $x_1, x_2$ in terms of $a, b, c$.

Since $\alpha, \beta$ are roots of $ax^2 + bx + c = 0$, we see that

$$\alpha + \beta = -b/a, \alpha \cdot \beta = c/a \ (*)$$

(we can assume $a \neq 0$ as otherwise $\alpha, \beta$ are easy to compute).

Note that $x_1 + x_2 = 2\alpha + 3\beta + 2\beta + 3\alpha = 5\alpha + 5\beta = 5(\alpha + \beta) = -5b/a$.

For $x_1x_2 = (2\alpha + 3\beta)(2\beta + 3\alpha) = 6(\alpha^2 + \beta^2) + 13\alpha\beta = 6[(\alpha + \beta)^2 - 2\alpha\beta] + 13\alpha\beta = 6(\alpha + \beta)^2 + 6\alpha\beta$.

Using relation (*) we get $x_1x_2 = 6(-b/a)^2 + c/a = \frac{6b^2 + ac}{a^2}$.

Since we computed the sum and the product of the numbers $x_1, x_2$ we can write now the equation that has them as solutions:

$$x^2 + \frac{5b}{a}x + \frac{6b^2 + ac}{a^2} = 0,$$

or after multiplying in both sides by $a^2$:

$$a^2x^2 + 5abx + 6b^2 + ac = 0.$$

(3) (5 points) Find the four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ as a product of two quadratic polynomials with integer coefficients.

Proof. We need to solve $z^4 = -4$, so let us write $-4$ in trigonometric form. The absolute value of $-4$ is 4, and the equations $\cos\theta = -4/4, \sin\theta = 0/4 = 0$ have $\theta = \pi$ as solution in $[0, 2\pi)$.

So, $-4 = 4(\cos(\pi) + i\sin(\pi)) = 4e^{i\pi}$.

The four roots of our equation are given by the formula:

$$z = 4^{1/4}\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right),$$

where $k = 0, 1, 2, 3$.

By explicitly computing we get the roots :

$$\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i, -1 + i, -1 - i, 1 - i.$$

So, $z^4 + 4$ factors as

$$z^4 + 4 = (z - (1 + i))(z - (-1 + i))(z - (-1 - i))(z - (1 - i)).$$
It remains to pair these factors to get a factorization that has only integer coefficients. Pair the first with the last and the middle two.

So, 
\[ z^4 + 4 = ((z - 1) - i^2)((z + 1)^2 - i^2) = ((z - 1)^2 + 1)((z + 1)^2 + 1) = (z^2 - 2z + 2)(z^2 + 2z + 2). \]

(4) (5 points) Show that \(|z| < 1\) implies that \(|Im(1 - \overline{z} + z^2)| < 3\).

Proof. Done in class. □

(5) (5 points) (graduate students) Consider \(a, b, c, m \in \mathbb{R}\) such that \(m > 1\). If

\[ \frac{a}{m + 1} + \frac{b}{m} + \frac{c}{m - 1} = 0, \]

show that \(b^2 - 4ac \geq 0\).

Proof. It either \(a\) or \(c\) is zero, then there is nothing to prove since \(b^2 \geq 0\).

So, let us assume that both \(a, c\) are nonzero.

Denote \(f(x) = ax^2 + bx + c\).

Note that \(f(0) = c\).

Compute now

\[ f(m/m + 1) = a\frac{m^2}{(m + 1)^2} + b\frac{m}{m + 1} + c = \frac{m^2}{m + 1}\left[\frac{a}{m + 1} + \frac{b}{m}\right] + c. \]

But

\[ \frac{a}{m + 1} + \frac{b}{m} + \frac{c}{m - 1} = 0, \]

so by substituting

\[ f(m/m + 1) = \frac{m^2}{m + 1}\left[-\frac{c}{m - 1}\right] + c = -\frac{cm^2}{m^2 - 1} + c = -\frac{c}{m^2 - 1}. \]

Compute now

\[ f(0)f(m/m + 1) = -\frac{c^2}{m^2 - 1} < 0, \]

since \(m > 1\).

This says that the equation \(f(x) = 0\) has a root between 0 and \(m/m + 1\) so it has two real roots, that is \(b^2 - 4ac \geq 0\). □