Undergraduate Mathematics Competition at Georgia State

Do any four out of the six problems. Indicate the four problems that you choose clearly.

(1) (10 points)
Each of the numbers \(a_1, \ldots, a_n\) is 1 or \(-1\). Assume that
\[
S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0.
\]
Show that 4 must divide \(n\).

Proof. List all the numbers among \(a_1, \ldots, a_n\) that are \(-1\).
One at a time, replace that number by 1 in the sum \(S\). Since each number appears in exactly four terms, the sum will be affected in four places. It will change by \(\pm 2\) for each of the four terms.
But \(\pm 2 \pm 2 \pm 2 \pm 2\) is a number divisible by 4.
Hence after we finish switching the negative numbers to positive ones, we modify our sum by a number divisible by four.
But, at the end when all numbers are equal to 1, the sum is \(n\) so 4 must be divisible by \(n\).

□

(2) (10 points)
Suppose that \(0 < x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\) and \(x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n = 1\).
Show that
(a) the function
\[
f(t) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{(1-t)x_i + ty_i}
\]
satisfies that \(f''(t) \geq 0\) for \(0 \leq t \leq 1\), and
(b) \[
\sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} \geq 0
\]
and equality holds if and only if \(x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n\).

Proof. As \(f'(t) = \sum_{i=1}^{n} x_i \frac{y_i - x_i}{(1-t)x_i + ty_i}\), \(f''(t) = \sum_{i=1}^{n} x_i \frac{(y_i - x_i)^2}{((1-t)x_i + ty_i)^2}\), \(f(0) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{x_i} = 0\), and \(f'(0) = \sum_{i=1}^{n} (x_i - y_i) = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i = 1 - 1 = 0\), we see that \(f''(t) \geq 0\) for \(0 \leq t \leq 1\), and \(f''(t) > 0\) for \(0 < t < 1\) if \(x_i \neq y_i\) for at least one value of \(i\).

Hence our function \(f\) is convex on \([0, 1]\), and as such its graph lies above the graph of any of its tangent lines. In particular, \(f(t) \geq f(0) + tf'(0)\) for \(0 \leq t \leq 1\), and \(f(t) > f(0) + tf'(0)\) for \(0 \leq t < 1\) if \(x_i \neq y_i\) for at least one value of \(i\). Thus \(f(1) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} \geq f(0) = 0\), and here strict inequality holds as soon as \(x_i \neq y_i\) for at least one \(i\).

□

(3) (10 points) Find the number of solutions of the equation
\[
\sin x = \frac{x}{100}.
\]
Proof. Since \( \sin \) takes values in \([-1, 1]\), all solutions of the equation are in the interval \([-100, 100]\). We first count the nonnegative solutions. As \( 31\pi < 100 < 32\pi \), the graph of \( \sin \) has between 0 and 100, 16 identical arches above the x axis. The graph of \( \frac{x}{100} \) intersects each of them twice. So, there are 32 nonnegative solutions, one being 0. Since the number of negative solutions equals the number of positive ones, the total number of solutions is 63.

\[ \square \]

(4) (10 points) Prove that for any set of \( n \) integers there is a subset of them whose sum is divisible by \( n \).

**Proof.** Let \( x_1, \ldots, x_n \) the \( n \) integers.

Write \( y_1 = x_1, y_2 = x_1 + x_2, \ldots, y_n = x_1 + \cdots + x_n \).

If one of the \( y_i \) is a multiple of \( n \) then we are done.

If not, then at least two of these \( n \) numbers must give the same nonzero remainder when dividing to \( n \).

Say those numbers are \( y_1, y_j, i < j \). So, \( n \) divides \( y_j - y_i \) and note that \( y_j - y_i \) is a sum as described in the statement of the problem.

\[ \square \]

(5) (10 points) Compute

\[
\frac{1}{2 + \sqrt{2}} + \frac{1}{3\sqrt{2} + 2\sqrt{3}} + \cdots + \frac{1}{(n + 1)\sqrt{n} + n\sqrt{n + 1}} = \sum_{k=1}^{n} \frac{1}{(k + 1)\sqrt{k} + k\sqrt{k + 1}}.
\]

**Proof.**

\[
\frac{1}{(k + 1)\sqrt{k} + k\sqrt{k + 1}} = \frac{1}{(k + 1)\sqrt{k} + k\sqrt{k + 1}} \cdot \frac{(k + 1)\sqrt{k} - k\sqrt{k + 1}}{(k + 1)\sqrt{k} - k\sqrt{k + 1}}.
\]

We get

\[
\frac{(k + 1)\sqrt{k} - k\sqrt{k + 1}}{(k + 1)^2k - k^2(k + 1)} = \frac{(k + 1)\sqrt{k} - k\sqrt{k + 1}}{k(k + 1)}.
\]

And this equals

\[
\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k + 1}}.
\]

So our sum equals

\[
\sum_{k=1}^{n} \left( \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) = 1 - \frac{1}{\sqrt{n+1}}.
\]

\[ \square \]

(6) (10 points)

Given \( n \) positive numbers \( a_1, \ldots, a_n \) such that \( a_1 = 1, a_n = 2 \) and \( a_k \leq \sqrt{a_{k-1}a_{k+1}} \) for \( k = 2, 3, \ldots, n - 1 \). Find \( \max_{1 \leq k \leq n} a_k \).

**Proof.** Let us show that if \( a_1, \ldots, a_n \) are positive, and \( a_k \leq \sqrt{a_{k-1}a_{k+1}} \) for \( k = 2, 3, \ldots, n - 1 \), then \( \max_{1 \leq k \leq n} a_k = \max \{a_1, a_n\} \), that is, in our case \( \max_{1 \leq k \leq n} a_k = 2 \).

Assume that \( \max_{1 \leq k \leq n} a_k = a_l, 1 < l < n, \ a_l > a_1, \ a_l > a_n \). Then by the condition

\[
a_l^2 \leq a_{l-1}a_{l+1} \leq \max_{1 \leq k \leq n} a_k \cdot \max_{1 \leq k \leq n} a_k = a_l^2,
\]

it follows that \( a_{l-1} = a_l = a_{l+1} \).

\[ \square \]
it follows that $a_{l-1} = \max_{1 \leq k \leq n} a_k = a_l$. Similarly, if $l - 1 > 1$, one gets $a_{l-1} = a_{l-2}$, etc. As the result, one has

$$a_l = a_{l-1} = a_{l-2} = \ldots = a_1.$$  

This contradicts the assumption $a_l > a_1$. □