Undergraduate Mathematics Competition at Georgia State
Do any four out of the six problems. Indicate the four problems that you choose clearly.

(1) (10 points)
Each of the numbers \(a_1, \ldots, a_n\) is 1 or \(-1\). Assume that
\[ S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0. \]
Show that 4 must divide \(n\).

(2) (10 points)
Suppose that \(0 < x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\) and \(x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n = 1\).
Show that
(a) the function
\[ f(t) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{(1-t)x_i + ty_i} \]
satisfies that \(f''(t) \geq 0\) for \(0 \leq t \leq 1\), and
(b) \[ \sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i} \geq 0 \]
and equality holds if and only if \(x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n\).

(3) (10 points) Find the number of solutions of the equation
\[ \sin x = \frac{x}{100}. \]

(4) (10 points) Prove that for any set of \(n\) integers there is a subset of them whose sum is divisible by \(n\).

(5) (10 points) Compute
\[ \frac{1}{2 + \sqrt{2}} + \frac{1}{3\sqrt{2} + 2\sqrt{3}} + \cdots + \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} = \sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}}. \]

(6) (10 points) Given \(n\) positive numbers \(a_1, \ldots, a_n\) such that \(a_1 = 1, a_n = 2\) and \(a_k \leq \sqrt{a_{k-1}a_{k+1}}\) for \(k = 2, 3, \ldots, n-1\). Find \(\max_{1 \leq k \leq n} a_k\).