DIVISORS OVER DETERMINANTAL RINGS DEFINED BY TWO BY TWO MINORS

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Let E and G be free modules of rank e and g, respectively, over a commutative noetherian ring R. The identity map on $E^* \otimes G$ induces the Koszul complex

$$\to \operatorname{Sym}_m E^* \otimes \operatorname{Sym}_n G \otimes \bigwedge^p (E^* \otimes G) \to \operatorname{Sym}_{m+1} E^* \otimes \operatorname{Sym}_{n+1} G \otimes \bigwedge^{p-1} (E^* \otimes G) \to$$

and its dual

$$\cdots \to D_{m+1}E \otimes D_{n+1}G^* \otimes \bigwedge^{p-1}(E \otimes G^*) \to D_m E \otimes D_n G^* \otimes \bigwedge^p (E \otimes G^*) \to \dots .$$

Let $\mathrm{H}_{\mathcal{N}}(m,n,p)$ be the homology of the top complex at $\operatorname{Sym}_m E^* \otimes \operatorname{Sym}_n G \otimes \bigwedge^p (E^* \otimes G)$ and $\mathrm{H}_{\mathcal{M}}(m,n,p)$ the homology of the bottom complex at $D_m E \otimes D_n G^* \otimes \bigwedge^p (E \otimes G^*)$. It is known that $\mathrm{H}_{\mathcal{N}}(m,n,p) \cong \mathrm{H}_{\mathcal{M}}(m',n',p')$, provided $m+m' = g-1$, $n+n' = e-1$, $p+p' = (e-1)(g-1)$, and $1-e \le m-n \le g-1$. In this talk we exhibit an explicit quasi-isomorphism M of complexes which gives rise to this isomorphism. The mapping cone of M is a split exact complex. Our complexes may be formed over the ring of integers; they can be passed to an arbitrary ring or field by base change.

Knowledge of the homology of the top complex is equivalent to knowledge of the modules in the resolution of the Segre module $\operatorname{Segre}(e,g,\ell)$, for $\ell = m-n$. The modules $\{\operatorname{Segre}(e,g,\ell) | \ell \in \mathbb{Z}\}$ are a set of representatives of the divisor class group of the determinantal ring defined by the 2×2 minors of an $e \times g$ matrix of indeterminants. If R is the ring of integers, then the homology $\mathrm{H}_{\mathcal{N}}(m,n,p)$ is not always a free abelian group. In other words, if R is a field, then the dimension of $\mathrm{H}_{\mathcal{N}}(m,n,p)$ depends on the characteristic of R. The module $\mathrm{H}_{\mathcal{N}}(m,n,p)$ is known when R is a field of characteristic zero; however, this module is not yet known over arbitrary fields.

The homology $\mathrm{H}_{\mathcal{N}}(m,n,p)$ is equal to the homology of the simplicial complex known as a chessboard complex with multiplicities. A chessboard complex is the matching complex of a complete bipartite graph.

The modules in the minimal resolution of the universal ring for finite length modules of projective dimension two are equal to modules of the form $\mathrm{H}_{\mathcal{N}}(m,n,p)$.