Spring 2009 Abstract Algebra II Homework-Assignment 1
Selected solutions.

(1) (5 points) Let H be a subgroup of G of finite index. Prove that H contains a normal subgroup of G of finite index.

(2) (5 points) Prove that in a group the subset of all elements that have only a finite number of conjugates is a subgroup.

(3) (5 points) If G is a finite group of order n and p is the smallest prime dividing $\mid G \mid$ then any subgroup of index p is normal.

Proof. Let H be a subgroup of index p in G. Then there exists a group homomorphism $G \to S_p$ with kernel equal to $N = \cap\, xHx^{-1}$. Of course, $N \subset H$.

So, G/N injects into S_p which means that $[G : N]$ divides $p!$.

But $[G : N] = [G : H][H : N] = p[H : N]$.

So, $[H : N]$ divides $(p-1)!$.

Note that $\mid G \mid= [G : H][H : N] \mid N \mid$ and since $[H : N]$ divides $(p-1)!$ we get that there exists a prime divisor of $\mid G \mid$ less than p unless $[H : N] = 1$.

So, $H = N$ which implies that H is a normal subgroup of G.

□

(4) (5 points) If the center of G has index n, prove that every conjugacy class has at most n elements.

Proof. Let $x \in G$. Then $Z(G) \subseteq N(x)$.

Note that the natural map $gZ(G)$ to $gN(x)$ is well-defined and surjective, so $[G : N(x)] \leq [G : Z(G)] = n$.

But $\mid C(x) \mid= [G : N(x)] \leq n$.

□

(5) (5 points) let $n \geq 4$, $\sigma = (12)(34)$. Show that the cardinality of the normalizer of σ in S_n is $8 \cdot (n-4)!$.

Proof. We need to count the number of elements which have the cycle decomposition equal to $(a\ b)(c\ d)$ in S_n.

The answer is $\frac{n(n-1)(n-2)(n-3)}{2 \cdot 2 \cdot 2}$.

But

$$\frac{n!}{\mid N(x) \mid} = \frac{n(n-1)(n-2)(n-3)}{8},$$

which implies the statement.

□