Spring 2009 Abstract Algebra II Homework-Assignment 2
Selected solutions.

(1) (5 points) Find an example of a finite group G that is not nilpotent but there exists H normal subgroup of G such that H and G/H are nilpotent.

Proof. Let $G = S_3$ and $H = A_3$. Then A_3 is a cyclic group of order 3 and it is nilpotent. G/H has two elements and so it is nilpotent as well.
S_3 is not nilpotent as it was mentioned in class.

□

(2) (5 points) Let H be a subgroup of S_n such that H is not contained in A_n. Show that exactly half the permutations in H are even.

Proof. Let σ be an odd permutation belonging to H.
Let $E = H \cap A_n$ and $O = H \cap (S_n \setminus A_n)$.
Let $f : E \to O$, $f(\alpha) = \sigma\alpha$.
Since σ is odd, then f is well-defined. It is also obviously injective.
To show surjectivity, let $\beta \in O$. Then $f(\sigma\beta) = \beta$ and $\sigma\beta$ is even.
So, E has as many elements as O.

□

(3) (5 points) Find all the conjugacy classes of D_4.

Proof. Let $G = D_4 = \{\sigma, \tau : \sigma^4 = e, \tau^2 = e, \sigma^3\tau = \tau\sigma\}$
Write the class equation for $G = D_4$:
$8 =\mid Z(G) \mid + \sum[G : N(x)]$.
Then since $Z(D_4) = \{e, \sigma^2\}$.
The conjugation class of σ is $\{\sigma, \tau\sigma\tau^{-1}\} = \{\sigma, \sigma^3\}$ (this is a hands on computation).
The rest of the conjugation classes are $\{\tau, \tau\sigma^2\}, \{\tau\sigma, \tau\sigma^3\}$ (also, a hands on calculation).
The class equation is : $8 = 2 + 2 + 2 + 2$.

□

(4) (5 points) Show that any group of order $4n + 2$ has a subgroup of index 2.
(5) (5 points) Consider G be the collection of all matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, c are real numbers. Show that $(G, \cdot)$ is solvable.