(1) (5 points) Let $R$ be a ring and $M$ be a left $R$-module and $m \in M$. Prove that $\text{Ann}(M) = \{ r : rM = 0 \}$ is a two-sided ideal and $\text{Ann}(m) = \{ r : r \cdot m = 0 \}$ is a left ideal in $R$.

(2) (5 points) Let $M$ be a left $R$-module. Compute $\text{Hom}_R(R, M)$. Compute $\text{Hom}_\mathbb{Z}(\mathbb{Z}^2, \mathbb{Z})$.

**Proof.** Let $f \in \text{Hom}_R(R, M)$. Then $f(r) = f(r \cdot 1) = rf(1)$, for all $r \in R$. So $f$ is completely determined by $f(1)$.

Now one can show that the map $T : \text{Hom}_R(R, M) \to M$, $T(f) = f(1)$ is an isomorphism of $R$-modules.

Note that if $(x, y) \in \mathbb{Z}^2$ then $(x, y) = x(1, 0) + y(0, 1)$.

So, if $f \in \text{Hom}_\mathbb{Z}(\mathbb{Z}^2, \mathbb{Z})$, then $f(x, y) = xf(1, 0) + yf(0, 1)$ and $f$ is hence completely determined by the pair of numbers $f(1, 0), f(0, 1)$.

Therefore, let us define $T : \text{Hom}_\mathbb{Z}(\mathbb{Z}^2, \mathbb{Z}) \to \mathbb{Z}^2$, by $T(f) = (f(1, 0), f(0, 1))$.

One can now prove that $T$ is $\mathbb{Z}$-linear, 1-1 and onto.

□

(3) (5 points) Compute $\text{Hom}_\mathbb{Z}(\mathbb{Z}_m, \mathbb{Z}_n)$.

(4) (5 points) Let $K, N, L$ be $R$-modules such that $K \oplus N = K \oplus L$ as $R$-modules, Show that $N \simeq L$.

**Proof.** Let $i : N \to K \oplus N, i(n) = (0, n)$ be the inclusion map which is $R$-linear.

Let $\pi : K \oplus L \to L, \pi(k, l) = l$, the natural projection which is also $R$-linear.

Using the fact that $K \oplus N = K \oplus L$, we can write $f = \pi i : N \to L$.

Note that $f$ is $R$-linear since it is the composition of two $R$-linear maps.

Let $f(n) = (0, n)$ with $k \in K$. So, $n = k = 0$. This shows that the kernel of $f$ is zero, so $f$ is 1 − 1.

Let $l \in L$. Then $(0, l) \in K \oplus N$, so $(0, l) = (k, n)$ for some $k \in K$, $n \in N$. But then $(0, n) = (−k, l)$.

Therefore $f(0, n) = l$, and $f$ is therefore onto.

□

(5) (5 points) Let $I$ be a left ideal of a ring $R$ such that $R/I$ is isomorphic to $R$ as $R$-module. Show that $I = Re$ where $e$ is idempotent (i.e., $e^2 = e$).

These three problems will count as extra-credit. Each of them is worth 2 points.

(6) Show that $\mathbb{Q}$ is isomorphic to $\text{End}_\mathbb{Z}(\mathbb{Q})$ as rings.

(7) Prove that $\mathbb{Q}$ is not free as $\mathbb{Z}$-modules.

(8) Let $R$ be a ring. Prove that if $R$ is completely reducible $R$-module then every $R$-module is completely reducible.