Spring 2009 Abstract Algebra II Homework-Assignment 4
Selected solutions. All rings are considered with unity.

(1) (5 points) Let R be a ring and M be a left R-module and $m \in M$. Prove that $\mathrm{Ann}(M) = \{r : rM = 0\}$ is a two-sided ideal and $\mathrm{Ann}(m) = \{r : r \cdot m = 0\}$ is a left ideal in R.

(2) (5 points) Let M be a left R-module. Compute $\mathrm{Hom}_R(R, M)$. Compute $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^2, \mathbf{Z})$.

Proof. Let $f \in \mathrm{Hom}_R(R, M)$. Then $f(r) = f(r \cdot 1) = rf(1)$, for all $r \in R$. So f is completely determined by $f(1)$.

Now one can show that the map $T : \mathrm{Hom}_R(R, M) \to M$, $T(f) = f(1)$ is an isomorphism of R-modules.

Note that if $(x, y) \in \mathbf{Z}^2$ then $(x, y) = x(1, 0) + y(0, 1)$.

So, if $f \in \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^2, \mathbf{Z})$, then $f(x, y) = xf(1, 0) + yf(0, 1)$ and f is hence completely determined by the pair of numbers $f(1, 0), f(0, 1)$.

Therefore, let us define

$$T : \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^2, \mathbf{Z}) \to \mathbf{Z}^2,$$

by $T(f) = (f(1, 0), f(0, 1))$.

One can now prove that T is $\mathbf{Z}$-linear, 1-1 and onto.

□

(3) (5 points) Compute $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}_m, \mathbf{Z}_n)$.

(4) (5 points) Let K, N, L be R-modules such that $K \oplus N = K \oplus L$ as R-modules, Show that $N \simeq L$.

Proof. Let $i : N \to K \oplus N, i(n) = (0, n)$ be the inclusion map which is R-linear.

Let $\pi : K \oplus L \to L, \pi(k, l) = l$, the natural projecion which is also R-linear.

Using the fact that $K \oplus N = K \oplus L$, we can write $f = \pi i : N \to L$.

Note that f is R-linear since it is the composition of two R-linear maps.

Let $f(n)$ =0 hence $(0, n) = (k, 0)$ with $k \in K$. So, $n = k = 0$. This shows that the kernel of f is zero, so f is $1 - 1$.

Let $l \in L$. Then $(0, l) \in K \oplus N$, so $(0, l) = (k, n)$ for some $k \in K$, $n \in N$. But then $(0, n) = (-k, l)$.

Therefore $f(0, n) = l$, and f is therefore onto.

□

(5) (5 points) Let I be a left ideal of a ring R such that R/I is isomorphic to R as R-module. Show that $I = Re$ where e is idempotent (i.e., $e^2 = e$).

These three problems will count as extra-credit. Each of them is worth 2 points.

(6) Show that $\mathbf{Q}$ is isomorphic to $\mathrm{End}_{\mathbf{Z}}(\mathbf{Q})$ as rings.

(7) Prove that $\mathbf{Q}$ is not free as $\mathbf{Z}$-modules.

(8) Let R be a ring. Prove that if R is completely reducible R-module then every R-module is completely reducible.