Spring 2009 Abstract Algebra II Homework-Assignment 5
Selected solutions.

(1) (5 points) Let R be a division ring. Prove that any module M over R is free. (Hint: can consult any text, please give a complete proof).

(2) (5 points) Prove that $\mathbb{Z}/p^n\mathbb{Z}$ is indecomposable for all $n > 0$ and p prime.

Proof. The list of all proper subgroups (hence $\mathbb{Z}$-submodules $\mathbb{Z}/p^n\mathbb{Z}$) is $X_k = p^k\mathbb{Z}/p^n\mathbb{Z}$, for $k = 1, ..., n-1$.

Note that $X_i \subset X_{i+1}$ so there are no proper subgroups X, Y such that $X \cap Y = 0$.

□

(3) (5 points) Let $V = \mathbb{R}^3$ and $\phi(x, y, z) = (2x + 3y + z, -4x - 2z, y + z)$ be a linear transformation of V.

Find the matrix of ϕ with respect to the canonical basis and then the one with respect to the basis $(1, 0, 0), (0, 1, 0), (1, 1, 1)$.

(4) (5 points) Let M, N be two free R-modules of finite rank, where R is commutative. Prove that $\text{Hom}(M, N)$ is free over R of rank $m \cdot n$.

Proof. Fix a basis for M and one for N. Let $m = rank(M)$, $n = rank(N)$. Each R-linear map can be described as a matrix in $M_{n,m}(R)$ and in fact we know that $\text{Hom}(M, N)$ is R-isomorphic with $M_{n,m}(R)$.

But $M_{n,m}(R)$ is isomorphic to $R^{n \cdot m}$ (just arrange your matrix in a long row of $n \cdot m$ elements) which is obviously free of rank $n \cdot m$.

□

(5) (5 points)

Let M be an R-module and $p \in \text{End}(M)$ idempotent. Prove that

$$M = Ker(p) \oplus Im(p).$$

Proof. Clearly $Ker(p) + Im(p) \subseteq M$.

Let $m \in M$ then $y = p(m) \in Im(p)$. Let $x = m - p(m)$. Note that $m = x + y$.

Remark that $x \in Ker(p)$ because $p(x) = p(m) - p^2(m) = p(m) - p(m) = 0$ (since $p = p^2$).

So, $M \subseteq Ker(p) + Im(p)$, as well. Hence $M = Ker(p) + Im(p)$.

Let $m \in Ker(p) \cap Im(p)$.

Then $m = p(a)$ and also $p(m) = 0$. Then $0 = p(m) = p(p(a)) = p^2(a) = p(a) = m$.

So $0 = Ker(p) \cap Im(p)$.

This finishes the proof.

□

These three problems will count as extra-credit. Each of them is worth 2 points.

(6) Let R be a ring and e an idempotent contained in the center of R.

Prove that $R \simeq Re \times R(1-e)$ as R-modules.

Prove that in fact Re, $R(1-e)$ are subrings of R with identity e, respectively $1-e$. Is the above homomorphism a ring homomorphism?

(7) For what n is $\mathbb{Z}/n\mathbb{Z}$ indecomposable over $\mathbb{Z}$?

(8) Let V be a K-vector space and X, Y be subspaces in V. Assume that V is finite dimensional over K. Prove that

$$\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y).$$