Spring 2009 Abstract Algebra II Homework-Assignment 5 **Name: ______________**

Write-up your solution carefully including all the details of the proof. Due Thursday, March 19.

Please staple your assignment. All rings are considered with unity.

(1) (5 points) Let R be a division ring. Prove that any module M over R is free. (Hint: can consult any text, please give a complete proof).

(2) (5 points) Prove that $\mathbb{Z}/p^n\mathbb{Z}$ is indecomposable for all $n > 0$ and p prime.

(3) (5 points) Let $V = \mathbb{R}^3$ and $\phi(x, y, z) = (2x + 3y + z, -4x - 2z, y + z)$ be a linear transformation of V.

Find the matrix of ϕ with respect to the canonical basis and then the one with respect to the basis $(1, 0, 0), (0, 1, 0), (1, 1, 1)$.

(4) (5 points) Let M, N be two free R-modules of finite rank, where R is commutative. Prove that $\mathrm{Hom}(M, N)$ is free over R of rank $m \cdot n$.

(5) (5 points)

Let M be an R-module and $p \in \mathrm{End}(M)$ idempotent. Prove that

$$M = Ker(p) \oplus Im(p).$$

These three problems will count as extra-credit. Each of them is worth 2 points.

(6) Let R be a ring and e an idempotent contained in the center of R.

Prove that $R \simeq Re \times R(1 - e)$ as R-modules.

Prove that in fact Re, $R(1 - e)$ are subrings of R with identity e, respectively $1 - e$. Is the above homomorphism a ring homomorphism?

(7) For what n is $\mathbb{Z}/n\mathbb{Z}$ indecomposable over $\mathbb{Z}$?

(8) Let V be a K-vector space and X, Y be subspaces in V. Assume that V is finite dimensional over K. Prove that

$$\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y).$$