Write-up your solution carefully including all the details of the proof. Due Tuesday, March 31.

Please staple your assignment. All rings are considered with unity.

(1) (5 points) Let $A, B$ be two $m \times n$ matrices with entries in a field $K$. Prove that

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

(2) (5 points) Let $A, B$ be two $n \times n$ matrices with entries in a field $K$. Prove that

$$\text{rank}(A \cdot B) \geq \text{rank}(A) + \text{rank}(B) - n.$$ 

(3) (5 points) Let $A$ be a matrix with real entries, Show that $\text{rank}(A) = \text{rank}(A^T A)$.

(4) (5 points) Compute the Smith normal form of the following matrix with integer entries:

$$\begin{pmatrix}
7 & 5 & 2 \\
3 & 3 & 0 \\
13 & 11 & 2
\end{pmatrix}$$

(5) (5 points)

Compute the Smith normal form of the following matrix with entries in $\mathbb{Q}[x]$:

$$\begin{pmatrix}
5 - x & 1 & -2 & 4 \\
0 & 5 - x & 2 & 2 \\
0 & 0 & 5 - x & 3 \\
0 & 0 & 0 & 4
\end{pmatrix}$$

These three problems will count as extra-credit. Each of them is worth 2 points.

(6) Let $R$ be a domain and $I$ an ideal of $R$ such that $I$ is $R$-free. Prove that $I$ is principal.

(7) Let $V = K^{(n)}$ be a vector space over a field $K$. Show that $V \simeq V \oplus V$ as $K$-vector spaces.

(8) Let $V = K^{(n)}$ be a vector space over a field $K$. Let $R = \text{End}_K(V)$.

Show that $R \simeq R \oplus R$ as $R$-right modules (the module action is the natural one).