

Write-up your solution carefully including all the details of the proof. Due Tuesday, March 31.

Please staple your assignment. All rings are considered with unity.

- (1) (5 points) Let  $A, B$  be two  $m \times n$  matrices with entries in a field  $K$ . Prove that

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

- (2) (5 points) Let  $A, B$  be two  $n \times n$  matrices with entries in a field  $K$ . Prove that

$$\text{rank}(A \cdot B) \geq \text{rank}(A) + \text{rank}(B) - n.$$

- (3) (5 points) Let  $A$  be a matrix with real entries, Show that  $\text{rank}(A) = \text{rank}(A^T A)$ .

- (4) (5 points) Compute the Smith normal form of the following matrix with integer entries:

$$\begin{pmatrix} 7 & 5 & 2 \\ 3 & 3 & 0 \\ 13 & 11 & 2 \end{pmatrix}$$

- (5) (5 points)

Compute the Smith normal form of the following matrix with entries in  $\mathbb{Q}[x]$ :

$$\begin{pmatrix} 5-x & 1 & -2 & 4 \\ 0 & 5-x & 2 & 2 \\ 0 & 0 & 5-x & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

These three problems will count as extra-credit. Each of them is worth 2 points.

- (6) Let  $R$  be a domain and  $I$  an ideal of  $R$  such that  $I$  is  $R$ -free. Prove that  $I$  is principal.  
(7) Let  $V = K^{(\mathbb{N})}$  be a vector space over a field  $K$ . Show that  $V \simeq V \oplus V$  as  $K$ -vector spaces.  
(8) Let  $V = K^{(\mathbb{N})}$  be a vector space over a field  $K$ . Let  $R = \text{End}_K(V)$ .  
Show that  $R \simeq R \oplus R$  as  $R$ -right modules (the module action is the natural one).