Spring 2009 Abstract Algebra II Homework-Assignment 7 **Name:** ______________

Write-up your solution carefully including all the details of the proof. Due Tuesday, March 31.

Please staple your assignment. All rings are considered with unity.

(1) (5 points) Let A, B be two $m \times n$ matrices with entires in a field K. Prove that

$$rank(A + B) \leq rank(A) + rank(B).$$

(2) (5 points) Let A, B be two $n \times n$ matrices with entires in a field K. Prove that

$$rank(A \cdot B) \geq rank(A) + rank(B) - n.$$

(3) (5 points) Let A be a matrix with real entries, Show that $rank(A) = rank(A^T A)$.

(4) (5 points) Compute the Smith normal form of the following matrix with integer entries:

$$\begin{pmatrix} 7 & 5 & 2 \\ 3 & 3 & 0 \\ 13 & 11 & 2 \end{pmatrix}$$

(5) (5 points)

Compute the Smith normal form of the following matrix with entries in $\mathbb{Q}[x]$:

$$\begin{pmatrix} 5-x & 1 & -2 & 4 \\ 0 & 5-x & 2 & 2 \\ 0 & 0 & 5-x & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

These three problems will count as extra-credit. Each of them is worth 2 points.

(6) Let R be a domain and I an ideal of R such that I is R-free. Prove that I is principal.

(7) Let $V = K^{(\mathbb{N})}$ be a vector space over a field K. Show that $V \simeq V \oplus V$ as K-vector spaces.

(8) Let $V = K^{(\mathbb{N})}$ be a vector space over a field K. Let $R = \mathrm{End}_K(V)$.

Show that $R \simeq R \oplus R$ as R-right modules (the module action is the natural one).